CALCULATION OF POTENTIAL FLOW
ABOUT 2D LIFTING SURFACES

by

Per-Age Krogstad

IIHR Report No. 243
Iowa Institute of Hydraulic Research
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SUMMARY

This report describes a method of calculating the inviscid flow about lifting 2D surfaces. The method has been designed particularly for use in predicting the lift characteristics of wings. However, the method should be applicable to any geometry if the rules for describing the geometry are followed.

The method is based on the well known surface panel methods. These approximate the surface by a series of straight panels. The flow around the body is then modelled by placing sources or sinks on these panels so that the velocity component normal to the surface at some control point vanishes. Therefore, at least locally around the control point, the surface will be a streamline.

The method has been applied to different standard airfoil shapes with good results. Also for more complex shapes the predictions come out satisfactorily.

From the experience obtained from the applications of the program it was found that if the only interest lies in predicting the integral variables, like the lift and pitching moment coefficients, satisfactory results may be obtained using less than 30 surface panels of rather arbitrary shapes. However, if the interest is focused on the actual pressure distribution, care must be taken in the distribution of the panels to prevent oscillations in the solution.
# TABLE OF CONTENTS

Summary........................................................................................................ii

1. Introduction.................................................................................................1

2. Flow Around Arbitrary 2D Surfaces..........................................................1
   2.1 Basic considerations...........................................................................1
   2.2 Mathematical formulation...................................................................4
   2.3 Evaluation of the influence integrals...................................................8
   2.4 The Kutta condition............................................................................10

3. Numerical Experiments.............................................................................11
   3.1 Non-lifting surfaces..........................................................................11
   3.2 Lifting surfaces..................................................................................12

4. Applications...............................................................................................14

5. Conclusions...............................................................................................17

6. References..................................................................................................18

7. Appendix A; Program Listing.................................................................20

Figures............................................................................................................27
1. INTRODUCTION

Panel methods have obtained increasing popularity during the last few decades and have now substituted analytical methods for most applications in aerodynamics. The main reason is of course the simplicity by which the panel method solutions may be obtained. (Basically the effort required to obtain the solution depends on how efficiently one can solve a $N \times N$ matrix). Secondly, with fast, high storage capacity computers, the surface may be discretized to give a sufficiently accurate description even of 3D surfaces so that the calculations may be performed to an accuracy comparable to analytical methods within reasonable computer time. Obviously for most 3D surfaces the complex geometry will not allow the use of analytical methods.

The ultimate goal of this work is to develop a method for 3D lifting surfaces. For the non-lifting case the extension from 2D to 3D is straightforward and only requires the use of 3D source and sink terms instead of the 2D expressions used here. A formulation of the 3D nonlifting case may be found in Mori and Nishimoto (1981) who calculated the 3D surface wave pattern around a ship. However, the extension to 3D lifting surfaces generates considerable logical problems related to the streamwise and lateral distribution of vorticity as pointed out by Hess (1972). We will therefore here only concentrate on solving the 2D lifting surface problem.

2. FLOW AROUND ARBITRARY 2D SURFACES

2.1 Basic Considerations. It will be assumed that the flow about any closed surface may be modelled by fundamental potential flows like uniform flows, sources/sinks and discrete vortex lines. The sources and sinks are distributed on a series of panels. Basically, two types of panels may be used, either mean line panels or surface panels. The use of mean line panels is particularly well suited for symmetric or thin bodies where the change in the velocity field caused by the presence of the body may be considered only as a perturbation of the free stream. In this case the sources and sinks are placed along the mean surface in the streamwise direction. The formulation for this problem may be found in Miloh and Landweber (1980).
However we will not restrict ourselves to thin or symmetric bodies and will therefore use surface panels. In this case the surface itself is approximated by a number of panels on which the sources and sinks are distributed. The panel size need not be fixed, so by adjusting the size according to some relevant parameter like for example the radius of surface curvature, the complexity of the body constitutes no real limitation. If the computed flow around the body is to be a good approximation to the inviscid solution the surface has to be a streamline. This implies that the flow is tangent to the surface everywhere or equivalently, that the velocity normal to the surface vanishes. This requires the distribution of source and sink strengths to be carefully adjusted. However, if the source strength along a panel is constant (which will be used here and which gives the simplest mathematical formulation) the number of adjustable constants is limited to the number of panels used. The point at which the normal velocity vanishes will be called a control point and we will allot one control point to each panel. This point will be located at the panel center. Obviously the approximation of the surface to a streamline will improve with increasing number of control points and in the limit when the number of points tend to infinity the exact solution should be obtained.

It is known from potential flow theory that for a closed body the sum of sources and sinks should be zero and this may be used as an independent check on the accuracy of the solution.

To produce lift it is necessary to introduce circulation in the flow field. According to the Kutta-Joukowsky theorem (see for example Bertin and Smith (1979)) the lift is given by

\[ L = \rho U_\infty \Gamma \]  \hspace{1cm} (2.1.1)

\( \Gamma \) being the circulation obtained by integrating the velocity vector along any closed path including all vortices. Because the path of integration is arbitrary (except for the direction which determines the sign of \( \Gamma \)) the distribution of \( \Gamma \) is of no importance when it comes to calculating the overall lift. However, the velocity distribution is affected by the vortex distribution so some care should be taken to assure a correct velocity distribution. In this work the vortices will be placed along the mean surface
which allows the largest possible distance between the control points and the singularity at the vortex center.
2.2. **Mathematical Formulation.** The physical problem will typically look like that shown in figure 2.2.1.

If \( O \) are the points defining the geometry of the surface we will let the straight lines between the \( O \) be the panels. (Note that the logic in the program requires the points to be entered clockwise starting at the trailing edge and returning to the exact same coordinate set. If this is not done an error message is given and the program halts).

On each panel sources or sinks of constant strength, \( k_j \), are distributed. The control point at which the normal velocity should be zero is taken as the midpoint. Therefore, if the number of coordinates entered is \( M + 1 \), \( M \) panels and control points will be generated. Thus

\[
\begin{align*}
X_c(m) &= \frac{1}{2} \left( X_p(m) + X_p(m+1) \right) \\
Y_c(m) &= \frac{1}{2} \left( Y_p(m) + Y_p(m+1) \right)
\end{align*}
\]

Discrete vortices will be placed midway between the upper and lower surface points except at the leading and trailing edge. Because the program assumes that for a given \( X \) position a coordinate is given for both the upper and the lower surface the number of vortices will be \( N = M/2 - 1 \). Hence
\[ X_v(n) = X_p(n+1) \]
\[ Y_v(n) = \frac{1}{2} (Y_p(n+1) + Y_p(M+1-n)) \]
\[ 1 \leq n \leq N \]  

(2.2.2)

Because the governing equation for potential flows is linear, the velocity potential at a general point, \( i \), is given as the sum of all the fundamental potentials in the flow field

\[
\phi_i = U_\infty (x_i \cos \alpha + y_i \sin \alpha) + \sum_{j=1}^{M} \frac{\Gamma_j}{2\pi} \int_{S_j} \ln r_{ij} \, ds
- \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi} \tan^{-1} \left( \frac{y_i - y_k}{x_i - x_k} \right)
\]

(2.2.3)

where the first term is the velocity potential from the free stream which is at an angle of attack, \( \alpha \), with respect to the positive x-axis. The second term is the sum of the contributions from all the source or sink terms. Because the panel has a finite length, \( S_j \), the distance from the source to the control point, \( r_{ij} \), varies along the panel so for each panel the potential influence at the control point must be calculated from an integral. The last term is the influence from all the vortices.

The velocities along the normal and tangent vectors to the panel \( i \) at its control point will be

\[
u_{ni} = \hat{\nu} \cdot \hat{n}_i
\]

(2.2.4a)

\[
u_{ti} = \hat{\nu} \cdot \hat{t}_i
\]

(2.2.4b)

With reference to figure 2.2.1 the normal and tangent vectors will be written

\[
\hat{n}_i = \hat{i} \cos \beta_i + \hat{j} \sin \beta_i
\]

(2.2.5a)

\[
\hat{t}_i = \hat{i} \cos \delta_i + \hat{j} \sin \delta_i
\]

(2.2.5b)

Combining equations (2.2.4) and (2.2.5) gives

\[
u_{ni} = \frac{\partial \phi_i}{\partial x_i} \cos \beta_i + \frac{\partial \phi_i}{\partial y_i} \sin \beta_i
\]

(2.2.6a)
\[ u_{ti} = \frac{\partial \phi_i}{\partial x_i} \cos \delta_i + \frac{\partial \phi_i}{\partial y_i} \sin \delta_i \] (2.2.6b)

Introducing the definition of \( \phi_i \) of equation (2.2.3) gives

\[ u_{ni} = u_\infty \cos (\alpha - \beta_i) \]

\[ + \sum_{j=1}^{M} \frac{1}{2\pi} \int_{S_j} \frac{(x_i-x_j)\cos \beta_i + (y_i-y_j)\sin \beta_i}{(x_i-x_j)^2+(y_i-y_j)^2} \, ds \]

\[ + \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi} \frac{(y_i-y_k)\cos \beta_i - (x_i-x_k)\sin \beta_i}{(x_i-x_k)^2+(y_i-y_k)^2} \] (2.2.7a)

\[ u_{ti} = u_\infty \cos (\alpha - \delta_i) \]

\[ + \sum_{j=1}^{M} \frac{1}{2\pi} \int_{S_j} \frac{(x_i-x_j)\cos \delta_i - (y_i-y_j)\sin \delta_i}{(x_i-x_j)^2+(y_i-y_j)^2} \, ds \]

\[ + \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi} \frac{(y_i-y_k)\cos \delta_i - (x_i-x_k)\sin \delta_i}{(x_i-x_k)^2+(y_i-y_k)^2} \] (2.2.7b)

Observe that these velocities are defined in the panel \( i \) normal and tangential coordinate system. Therefore if the panel is on the upper surface, a positive tangential velocity is pointing in the opposite direction as for the corresponding panel on the lower surface. To bring the result over in the \( x-y \) coordinate system the coordinate transformation

\[ u_{xi} = u_{ti} \cos \delta_i - u_{ni} \sin \delta_i \] (2.2.8a)

\[ u_{yi} = u_{ti} \sin \delta_i + u_{ni} \cos \delta_i \] (2.2.8b)

must be used. Of course for the solution obtained \( u_{ni} = 0 \).

To simplify expressions (2.2.7) we will introduce a shorthand notation for the integrals and sums.
\[ I_{ij} = \int_{S_j} \frac{(x_i-x_j)\cos\phi_i + (y_i-y_j)\sin\phi_i}{(x_i-x_j)^2 + (y_i-y_j)^2} \, ds \quad (2.2.9a) \]

\[ F_{ij} = \int_{S_j} \frac{(x_i-x_j)\cos\delta_i + (y_i-y_j)\sin\delta_i}{(x_i-x_j)^2 + (y_i-y_j)^2} \, ds \quad (2.2.9b) \]

\[ I_{iM+1} = \sum_{k=1}^{N} \frac{\Gamma_k}{\Gamma_0} \frac{(y_i-y_k)\cos\phi_k - (x_i-x_k)\sin\phi_k}{(x_i-x_k)^2 + (y_i-y_k)^2} \quad (2.2.10a) \]

\[ F_{iM+1} = \sum_{k=1}^{N} \frac{\Gamma_k}{\Gamma_0} \frac{(y_i-y_k)\cos\delta_k - (x_i-x_k)\sin\delta_k}{(x_i-x_k)^2 + (y_i-y_k)^2} \quad (2.2.10b) \]

where \( I_{ij} \) and \( F_{ij} \) give the contribution from the source on panel \( j \) to the normal and tangential velocities respectively at control point \( i \). These, which may be denoted influence integrals, depend solely on the geometry of the body and are independent of angle of attack. Therefore, once these have been calculated, they may be stored to be used to find the solution at any angle of attack. The terms \( I_{iM+1} \) and \( F_{iM+1} \) give the total influence from all vortices on the normal and tangential velocities. The reason for writing these terms in this manner will become clear in section 3.2 but it is immediately seen that if \( \Gamma_k = \Gamma_0 \) these influence parameters too will depend only on geometry.

Introducing (2.2.9) and (2.2.10) in (2.2.7) this becomes

\[ u_{ni} = u_\infty \cos (\alpha - \beta_i) + \sum_{j=1}^{M} \frac{k_j}{2\pi} I_{ij} + I_{iM+1}\Gamma_0 \quad (2.2.11a) \]

\[ u_{ti} = u_\infty \cos (\alpha - \delta_i) + \sum_{j=1}^{M} \frac{k_j}{2\pi} F_{ij} + F_{iM+1}\Gamma_0 \quad (2.2.11b) \]
2.3. Evaluation of the Influence Integrals. To calculate the integrals let us introduce a new coordinate system \((\xi, \eta)\) locally at the panel \(j\) along which the integral is to be taken

\[
\delta_j
\]

![Diagram](image)

Fig. 2.3.1

If \(\mathbf{n}\) is the normal vector and \(\mathbf{t}\) the tangent vector of the panel, \((x_1, y_1)\) will be the coordinate point defining that panel, having the lowest index. (For the first panel the trailing edge will be given as \((x_1, y_1)\) whereas for the last panel the trailing edge is given by \((x_2, y_2)\).) The new coordinate system is placed with its origin at \((x_1, y_1)\) with \(\mathbf{t}\) pointing along \(\mathbf{n}\). Therefore, if \(\delta_j\) is the angle between the \(\mathbf{t}\) vector and the \(x\)-axis the following coordinate transformation applies

\[
x_j = x_1 + \xi \cos \delta_j
\]

\[y_j = y_1 + \xi \sin \delta_j \quad (2.3.1a)
\]

\[
\xi = (x_1 - x_j) \cos \beta_1 + (y_1 - y_j) \sin \beta_1 + \frac{f_2}{2} \cos (\delta_j - \beta_1)
\]

Introducing (2.3.1) into (2.2.9) \(I_\xi\) and \(F_\xi\) may be integrated giving

\[
I_\xi = \frac{2}{\sqrt{g}} \left\{ (x_1 - x_j) \cos \beta_1 + (y_1 - y_j) \sin \beta_1 + \frac{f_2}{2} \cos (\delta_j - \beta_1) \right\}
\]

\[
= \tan^{-1} \left( \frac{2s + f_2}{\sqrt{g}} \right) - \tan^{-1} \left( \frac{f_2}{\sqrt{g}} \right) - \frac{1}{2} \cos (\delta_j - \beta_1) \ln \frac{f_1 + f_2 s + s^2}{f_1} \quad (2.3.2a)
\]

\[
F_\xi = \frac{2}{\sqrt{g}} \left\{ (x_1 - x_j) \cos \delta_i + (y_1 - y_j) \sin \delta_i + \frac{f_2}{2} \cos (\delta_j - \delta_i) \right\}
\]

\[
= \tan^{-1} \left( \frac{2s + f_2}{\sqrt{g}} \right) - \tan^{-1} \left( \frac{f_2}{\sqrt{g}} \right) - \frac{1}{2} \cos (\delta_j - \delta_i) \ln \frac{f_1 + f_2 s + s^2}{f_1} \quad (2.3.2b)
\]
where \( s \) is the length of the panel and

\[
\begin{align*}
  f_1 &= (x_i - x_1)^2 + (y_i - y_1)^2 = r_{11}^2 \\
  f_2 &= 2[(x_i - x_i) \cos \delta_j + (y_i - y_i) \sin \delta_j] \\
  g &= 4f_1 - f_2^2
\end{align*}
\]

(2.3.3a) (2.3.3b) (2.3.3c)

It should be pointed out that although the expressions in equation (2.3.2) may seem complicated they only differ by \( I_{ij} \) containing \( \beta_i \) whereas \( F_{ij} \) contains \( \delta_i \). Using multi-dimensional arrays this makes the programming of \( I_{ij} \) and \( F_{ij} \) very simple.

Special care must be taken to evaluate \( I_{ii} \) and \( F_{ii} \), i.e., the influence on the control point from its own panel because the integrals in this case are singular. Consider the problem illustrated in figure 2.3.2. The control point

![Diagram](image)

Fig. 2.3.2

is moved a small distance \( y \) away from the surface. With the coordinate system located at the panel midpoint the panel extends from \(-x/2\) to \(+x/2\). The velocity induced at \( i \) from a small section of the panel is directed along the \( \vec{v} \) vector so the total induced velocity along the \( y \) axis is given by

\[
v = \int_{-x/2}^{x/2} \frac{k_i}{2\pi r} \sin \theta \, dx
\]

(2.3.4a)
and the total induced velocity parallel to the x-axis will be

\[ u = \int_{-x/2}^{x/2} \frac{\cos \theta}{2\pi r} \, dx \]  \hspace{1cm} (2.3.4a)

During the integration y is kept constant so (2.3.4) becomes

\[ v = \frac{k_j y}{2\pi} \int_{-x/2}^{x/2} \frac{1}{y^2 + x^2} \, dx = \frac{k_j y}{2\pi} \left[ \tan^{-1} \left( \frac{x}{y} \right) \right]_{-x/2}^{x/2} \]  \hspace{1cm} (2.3.5a)

\[ u = \frac{k_j x}{2\pi} \int_{-x/2}^{x/2} \frac{x}{y^2 + x^2} \, dx = \frac{k_j x}{4\pi} \left[ \log(x^2 + y^2) \right]_{-x/2}^{x/2} \]  \hspace{1cm} (2.3.5b)

The interest now lies in the limiting case for which \( y \to 0 \) while keeping the integration limits fixed. This yields

\[ v = \frac{k_j y}{2} \]  \hspace{1cm} (2.3.6a)

\[ u = 0 \]  \hspace{1cm} (2.3.6b)

from which it is concluded that

\[ I_{ij} = \pi \]  \hspace{1cm} (2.3.7a)

\[ F_{ij} = 0 \]  \hspace{1cm} (2.3.7b)

### 2.4 The Kutta Condition.
Because the number of control points equals the number of source strengths, \( k_j \), to be determined another criterion must be supplied to determine \( \Gamma_0 \). Because the interest is connected to airfoil type of lifting surfaces the trailing edge will be a sharp discontinuity. Therefore, only one physical solution to the inviscid problem exists, namely when the pressure on the upper and lower surface at the trailing edge are equal. For any other solution the flow will have to turn around the trailing edge from the high pressure to the low pressure side giving a pressure at the trailing edge that tends to \(-\infty\). Therefore we will require that the pressure at the upper and lower side near the trailing edge tend to the same limit or equivalently that the tangential velocities on both surfaces are the same. This is known as the Kutta condition.
Because the calculations are only carried out from the first to the last control points the condition will have to be applied somewhat upstream of the trailing edge. However, Hess (1972) showed that the solution changes very little when the condition is applied between \( x/c = 0.98 \) and 1.00. Therefore the Kutta condition will be applied to the first and last control points. From equation (2.2.11b) we get

\[
\begin{align*}
  \frac{u_\infty \cos (\alpha - \delta_M)}{2\pi} + \sum_{j=1}^{M} \frac{k_j}{2\pi} F_{Mj} + F_{MM+1} \Gamma_0 \\
  = \frac{u_\infty \cos (\alpha - \delta_1)}{2\pi} + \sum_{j=1}^{M} \frac{k_j}{2\pi} F_{1j} + F_{1M+1} \Gamma_0
\end{align*}
\]

(2.4.1)

(Observe that for blunt bodies the Kutta condition is meaningless and therefore \( \Gamma_0 \) cannot be determined).

### 3. NUMERICAL EXPERIMENTS

#### 3.1. Non-Lifting Surfaces.

To make sure that the source/sink distribution is solved correctly a simple non-lifting geometry should first be computed. For this purpose the circular cylinder was chosen because the analytical solution for this is known. The cylinder was represented by first 8 and later 32 panels and the solution compared to the analytical solution

\[
u_t = 2u_\infty \sin \theta
\]

(3.1.1)

It was found that although the representation of the surface by only 8 panels leaves much to be desired (figure 3.1.1) the solution obtained was exact to five digits. However, although the surface representation using 32 panels is much better the solution was only accurate to four digits in the region of highest velocity \( \theta=90^\circ \). It is believed that this is due to numerical error buildup in the inversion of the much larger matrix.

Because the body considered is a closed body the net source and sink fluxes should balance. This was achieved to within \( 10^{-7} \) of the total source strength for both cases. From this exercise it was concluded that the source and sink distribution is solved satisfactorily.
As a further check on the flow around non-lifting bodies the flow around a symmetric airfoil at zero angle of attack was computed to see the effect of thickness distribution. For this purpose the NACA 0012 airfoil section was chosen (figure 3.1.2) which is a symmetric airfoil with 12% maximum thickness located at \( x/c = 0.3 \). The result is shown in figure 3.1.3 and the computations are compared to the solution given in Abbott and Von Doenhoff (1949) by conformal mapping. The agreement is seen to be quite good. However, the balance between source and sink terms in this case was reduced to \( 10^{-4} \) of the total source strength.

**3.2 Lifting Surfaces.** The next problem to be solved would be a lifting airfoil. As stated in Section 2.1 the lift may be calculated from the Kutta-Joukowski theorem and should therefore be independent of the way the vorticity is distributed inside the airfoil section. To check this statement the flow around the NACA 1408 airfoil was calculated using different vortex distributions. The NACA 1408 airfoil is shown in figure 3.2.1 and has maximum camber of 1% located at \( x/c = 0.4 \) and a maximum thickness of 8% located at \( x/c = 0.3 \).

The simplest case would be to use only a single vortex i.e.,

\[ \Gamma_k = 0 \text{ for } 1 < k < N, k \neq K \]  

(3.2.1)

Because the aerodynamic center is always located close to \( x/c = 1/4 \) the vortex was placed here on the camber line. The predicted lift coefficient as function of the angle of attack is shown in figure 3.2.2 and compared to the measurements given in Abbott and Von Doenhoff (1949). The lift coefficient at \( \alpha = 0^\circ \) is predicted somewhat high and at \( \alpha = 8^\circ \) it is about 20% too low. According to thin wing theory the slope of the lift curve should be \( 2\pi/\text{rad} \). The experimental slope is \( 1.95\pi/\text{rad} \) but the calculated value is only \( 1.46\pi/\text{rad} \). Also, as was to be expected, the velocity distribution shown in figure 3.2.3 for \( \alpha = 0^\circ \) is not very realistic, showing large velocity peaks close to the vortex center. It is therefore concluded that a single vortex is insufficient both for the calculation of lift and velocity distribution.

Next, the calculation was done letting all vorticities have the same strength, i.e.,
\[ \Gamma_k = \Gamma_0 \text{ for } 1 < k < N \] (3.2.2)

The lift curve for this vortex distribution is shown in figure 3.2.4. The predictions are seen to follow the experimental data very well and the predicted lift curve slope was 1.89\(\pi\)/rad. However an inspection of the surface velocity distribution plotted in figure 3.2.5 shows that a series of wiggles exist. The amplitude of the wiggles are seen to increase as the trailing edge is approached. Because the vortices all are of equal strength the velocity induced at the surface increases as the distance to the surface decreases. Therefore it was assumed that a better velocity distribution would be obtained if the vortex strength was made proportional to the local airfoil thickness

\[ \Gamma_k = \Gamma_0 (Y_p(M-k) - Y_p(k+1)) \text{ for } 1 < k < N \] (3.2.3)

In this way the velocity induced at the surface from the nearest vortex would be independent of \(x/c\) while, in order to satisfy the Kutta condition, the total circulation and therefore the section lift should remain approximately the same.

The effect on the lift curve was a slight increase in the lift curve slope to 2.05\(\pi\)/rad which is only 2.5% higher than the theoretical value (figure 3.2.6). However, the wiggles on the surface velocity distribution have been strongly reduced (figure 3.2.7).

According to a study by Hess (1972) the wiggles on the surface velocity distribution will only disappear if a continuous distribution of surface vorticity is used. From the experience gained in applying the program it was observed that the wiggles seem to be related to places where there is a considerable change in panel size. To check this, calculations were repeated for the same airfoil section but letting the size form a geometric progression so that neighboring panels increase or decrease by a fixed amount

\[ \frac{\Delta x_{n+1}}{\Delta x_n} = k \] (3.2.4)

where \(k\) was chosen to be 1.25 from the leading edge to \(x/c = 0.6\) and 1/\(k\) from \(x/c = 0.6\) to the trailing edge. \(\Delta x\) was chosen so that the total number of panels was kept constant. The result is shown in figure 3.2.8 and it may be
seen that all wiggles except one have disappeared. (Note that this time the pressure coefficient, $C_p = 1 - (u/u_\infty)^2$, was plotted which amplifies any wiggles present). This wiggle is related to the change from increasing to decreasing panel size where the rate of change of size is discontinuous. Therefore, to avoid wiggles the panels should be distributed so that the rate of change of panel size is continuous. One such possibility is to distribute the panels according to

$$\frac{\Delta x_{n+1}}{\Delta x_n} = k_0 + \frac{\partial k}{\partial x} x_n$$

(3.2.5)

where $\partial k/\partial x$ is chosen such that the panel size increases along the forward part and decreases at the rear. For the calculation shown in figure 3.2.9, $k_0$ was chosen to be 1.4, $\partial k/\partial x = -2/3$ which gives a crossover from increasing to decreasing panel size at $x/c = 0.6$ and $\Delta x_1 = 0.00175$. This gives a panel size which increases from 0.175% chord at the leading edge to about 10% at the mid section and then decreases back to about 5% at the trailing edge. It is seen that the pressure distribution is perfectly smooth and this was obtained without changes in the section lift coefficient. (The kink at the trailing edge is a graphical shortcoming as the solution tends to $C_p = 1$ at the trailing edge).

4. APPLICATIONS

To find out the validity of the solutions produced by the method some comparisons with reliable experiments should be done. Unfortunately, only a limited number of experiments were available at the time of writing, some of which were of doubtful accuracy. However, enough information was available to assess the general behavior of the method.

The integral variables like the lift and pitching moment coefficients were calculated assuming linear variation of the pressure coefficient between the control points. This was found to improve the predictions of these variables considerably, compared to the usual assumption of constant properties on the panels. Two NACA profiles were picked randomly and the predictions compared to the data available in Abbott and von Doenhoff (1949). Figure 4.1 shows that for the NACA 0012 profile, which was shown in figure 3.1.2, the pitching moment is predicted to be zero which is in
agreement with thin airfoil theory as well as the measurements. The symmetry condition is also satisfied by producing zero lift at zero angle of attack. As stated in section 3.2 the slope of the curve is somewhat higher than $2\pi$ which thin airfoil theory predicts, but still very close to the experimental value.

The other NACA profile for which the integral variables are shown is the NACA 4412. The profile, shown in figure 4.2, is a highly cambered airfoil with maximum camber of 4% at $x/c = 0.4$ and with 12% thickness. The effect of camber is to increase the lift at a given angle of attack and produce a pitch down moment around the quarter cord. Both effects are very clearly reproduced by the method and the agreement with the measurements is good, (figure 4.3). The pressure distribution at zero angle of attack is shown in figure 4.4.

From this exercise it was concluded that the integral parameters are satisfactorily predicted. However, good agreement for integral parameters does not guarantee correct behavior of the pressure distribution, which has to be checked separately.

Figure 4.5 and 4.6 show comparisons with the measurements of Pinkerton (1936). The measurements were taken at the midspan section of a three-dimensional wing of aspect ratio 6 without sweep or twist. The profile used was the NACA 4412 shown in figure 4.2. According to finite wing theory (see for example Clancy (1975)) the effective angle of attack changes with the aspect ratio as

$$\alpha_{\text{eff}} = \alpha_{\text{geom}} - \frac{C_L}{\pi AR}$$

(4.1)

so that to produce a certain lift coefficient the geometric angle of attack needed increases as the aspect ratio (AR) decreases.

From the very detailed measured pressure distribution (54 pressure taps were distributed along the chord line) the lift coefficient could be calculated and the equivalent two-dimensional angle of attack estimated. The potential flow calculated at this effective angle of attack is compared to the measurements in figure 4.5 and 4.6 and it is seen that the agreement is quite good with the relevant features correctly reproduced by the calculations. However, at the angle of attack of 2 degrees a level shift exist between the measured and computed pressure coefficients on the lower surface. This
resulted in the calculated lift coefficient being somewhat smaller than the value given by the experimental data. At the higher angle of attack strong viscous effects may be seen to influence the experimental data since there is no tendency towards a pressure recovery at the trailing edge producing a large pressure difference between the upper and lower surface. Except for this region the agreement is good.

Because the calculations of the symmetric airfoil shown in figure 4.4 gave good agreement with the theoretical distribution it was thought that the calculation of the measurements taken by Novak (1981) should come out fairly accurate. Novak did his measurements on a 30 degrees swept wing spanning the entire wind tunnel section. This was done to simulate an infinite swept wing for which no spanwise changes occur. The airfoil section was NACA 0012 taken normal to the leading edge. Therefore, the section seen by the air flow would be a profile with the same thickness but with a streamwise distance stretched as

\[ x^* = x / \cos(30) \]  \hspace{1cm} (4.2)

The result is shown in figure 4.7. It is seen that the experiment indicates a suction peak around \( x/c = 0.2 \) whereas the calculated peak lies at \( x/c = 0.09 \). Also, a level shift is present from \( x/c = 0.2 \) down to the trailing edge. A check on the experimental data revealed that there was a considerable spanwise pressure gradient so that the infinite swept wing assumption was not fulfilled. In fact, it is known from other experiments (van den Berg and Elsenaar (1972)) that infinite swept wing conditions are extremely difficult if not impossible to obtain in a wind tunnel test. This lack of infinite wing conditions possibly combined with some blockage effects are believed to be the cause of the bad agreement.

Somewhat better results were obtained in the calculations of the other set of wing data available at the Institute. The wing, shown in figure 4.8, is a so called undercambered wing originally designed for transonic applications. The wing was used by Ramaprian, Patel, and Sastry (1981) to produce an asymmetric turbulent wake. Because of the undercamber it was found necessary to use at least five panels on each surface from \( x/c = 0.8 \) to the trailing edge giving a total number of 56 panels for the section. The
result is shown in figure 4.9. The best agreement is obtained on the forward part whereas some deviation exists aft, possibly indicating some boundary layer effects. Once again the general behavior of the pressure field is satisfactorily predicted.

The last profile computed is the high lift profile shown in figure 4.10. This profile was designed for maximum lift in an inverse manner, i.e., by specifying the velocity distribution and solve for the corresponding geometry (see Liebeck and Ormsbee (1970)). In a later report Liebeck (1973) shows experimental data verifying the high performance of the section with lift coefficients as high as 2.2 and drag coefficients less than 0.008, giving a lift to drag ratio of almost 300. Unfortunately the coordinates for the profile were not given. However, the coordinates of another of the high lift sections, the Liebeck #33, was found in Walters and Wroniak (1974). Unfortunately their measurements were not of high quality and do not compare very well with the predictions as seen in figure 4.11. However, because the experimental values of Liebeck (1973) coincides almost exactly with his potential flow calculations it is believed that the analytical solution given by Walters and Wroniak is more correct than their experimental data. A comparison with their theoretical distribution is shown in figure 4.12. 58 panels were used, which is the highest number of panels used in the calculations shown in this report, in order to get a good representation of the leading and trailing edge regions as well as the rather sharp corner located on the upper surface at x/c = 0.28. For some unknown reason this corner is not reproduced in the solution of Walters and Wroniak until x/c = 0.39. (Their experimental data do however show the effect of the corner at the right position).

5. CONCLUSIONS

A calculation method has been described that is capable of calculating the potential flow around complex closed single surfaces with a sharp trailing edge or for which the downstream stagnation point is known. (A listing of the program is included in Appendix A).

It has been shown that the method behaves well both for lifting and nonlifting surfaces where all relevant features of the solution are
predicted. It is believed that the obtained solutions are of sufficiently high quality to be used for example as input to boundary layer calculations.

Somewhat disappointing was the finding that considerable attention must be given to the discretization of the body for which the solution is to be found. It proved essential to concentrate panels in the area where the geometry changed rapidly, i.e., near the leading and trailing edges. Typically the panels close to the leading edge would be \( \Delta x/c = 0.002 \) and near the trailing edge \( \Delta x/c = 0.04 \). It was found that in order to obtain a smooth velocity distribution the panel size should be distributed in a continuous manner, for example according to the formula given in equation (3.2.5), for which good solutions were always obtained.

Unfortunately the number of good experimental data available was rather limited. It is suggested that further checks with experiments are performed before the method is extended to fully 3D calculations.

6. REFERENCES


Pinkerton, R.M. "Calculated and Measured Pressure Distributions over the Midspan Section of the NACA 4412 Airfoil", Report 563, NACA, 1936.


APPENDIX A

Program Listing
%INSERT SYSCOM>ERROR.D.F
%INSERT SYSCOM>KEYS.F
%INSERT SYSCOM>ATKEYS
C AIRFOIL SECTION
C
C THIS PROGRAM CALCULATES THE FLOW AROUND AN AIRFOIL USING
C SOURCES AND SINKS DISTRIBUTED ON THE SURFACE PLUS A
C VORTEX LOCATED AT X/C=0.25 TO SATISFY THE KUTTA CONDITION.
C AN OPTION TO DISTINGUISH BETWEEN LIFTING AND NON-LIFTING
C SECTIONS IS AVAILABLE.
C
NP  =NUMBER OF SURFACE COORDINATES
ILIFT =LIFT PARAMETER, ILIFT=0 FOR NON-LIFTING SECTION
      ILIFT=1 FOR LIFTING SECTION
ALFA =ANGLE OF ATTACK WITH RESPECT TO CORDLINE
C =CORD LENGTH (BASIS FOR NON-DIMENSIONALIZATION)
XP =CORDWISE COORDINATE
YP =VERTICAL COORDINATE
THE COORDINATES SHOULD BE GIVEN GOING CLOCKWISE
AROUND THE SECTION STARTING AT THE TRAILING EDGE.
C
C IMPLICIT DOUBLE PRECISION(A-H,O-Z)
DIMENSION XP(70),YP(70),XC(69),YC(69),S(69),DEL(69,2),
F(70,70,2),CI(2),SI(2),H(70),SOR(69),UT(69),UX(69),UY(69),
,TEXT(20),XV(34),YV(34),CP(69)
CALL OPEN$A(A$WRIT+A$SAMF,'FOIL',4,2)
PI=3.1415926
PI2=PI/2.
C
C INPUT
C
CALL OPEN$A(A$READ+A$SAMF,'CORD',4,1)
READ(5,4) (TEXT(J),J=1,20)
READ(5,1)NP,ILIFT,ALFA,C
ALFA=ALFA/180.*PI
READ(5,2) (XP(J),J=1,NP)
READ(5,2) (YP(J),J=1,NP)
CALL CIOS$A(1)
1 FORMAT(2I3,2E10.3)
2 FORMAT(5E10.3)
4 FORMAT(20A2)
IF(XP(1).NE.XP(NP))STOP 2
IF(YP(1).NE.YP(NP))STOP 2
DO 3 J=1,NP
XP(J)=XP(J)/C
3 YP(J)=YP(J)/C
C
C CALCULATE CONTROL POINTS AND SURFACE SLOPES
C DEL(J,1)=PANEL SLOPE ANGLE WITH RESPECT TO THE CORDLINE
C DEL(J,2)=ANGLE OF OUTWARD NORMAL TO THE SURFACE

NP1=NP-1
DO 10 J=1,NP1
XC(J)=(XP(J)+XP(J+1))/2.
YC(J)=(YP(J)+YP(J+1))/2.
DX=XP(J+1)-XP(J)
DY=YP(J+1)-YP(J)
S(J)=DSQRT(DX**2+DY**2)
IF(XP(J+1).NE.XP(J))GO TO 14
IF(J.EQ.NP1)GO TO 15
IF(XP(J+2).GT.XP(J+1))DEL(J,1)=PI2
IF(XP(J+2).LT.XP(J+1))DEL(J,1)=-PI2
GO TO 10
15 IF(XP(J-1).LT.XP(J))DEL(J,1)=-PI2
IF(XP(J-1).GT.XP(J))DEL(J,1)=PI2
GO TO 10
14 DEL(J,1)=(YP(J+1)-YP(J))/(XP(J+1)-XP(J))
DEL(J,1)=DATAN(DEL(J,1))
IF(XP(J+1).LT.XP(J))DEL(J,1)=DEL(J,1)+PI
10 DEL(J,2)=DEL(J,1)+PI2
C CALCULATE THE INFLUENCE MATRIX
C
DO 11 I=1,NP1
DO 12 K=1,2
DA=DEL(I,K)
IF(K.EQ.1.AND.DA.GT.PI2)DA=DA+PI
CI(K)=DCOS(DA)
12 SI(K)=DSIN(DA)
DO 11 J=1,NP1
IF(I.NE.J)GO TO 13
F(I,J,1)=PI
F(I,J,2)=0.
GO TO 11
13 X1=XC(I)-XP(J)
Y1=YC(I)-YP(J)
CDJ=DCOS(DEL(J,1))
SDJ=DSIN(DEL(J,1))
F1=X1*X1+Y1*Y1
F2=-2.*(X1*CDJ+Y1*SDJ)
Q=4.5*F1-F2*F2
IF(Q.LT.0.)STOP 3
Q=DSCRT(Q)
TT=DATAN((2.*S(J)+F2)/Q)-DATAN(F2/Q)
TL=DLGC((F1+(F2+S(J))*S(J))/F1)
DO 11 K=1,2
L=3-K
DA=DEL(I,L)
IF(K.EQ.2.AND.DA.GT.PI2)DA=DA+PI
A=X1*CI(L)+Y1*SI(L)+F2/2.*DCOS(DEL(J,1)-DA)
B=0.5*DCOS(DEL(J,1)-DA)
F(I,J,K)=2./Q*X*TT-B*TL
11 CONTINUE
C CALCULATE INFLUENCE FROM VORTEX
C
SUMB=0.0
NVTEX=(NP-3)/2.
DO 20 J=1,NVTEX
XV(J)=XP(J+1)
YV(J)=(YP(J+1)+YP(NP-J))/2.
DBV=YV(J)-YP(J+1)
20 SUMB=SUMB+DBV
DO 21 I=1,NP1
F(I,NP,1)=0.0
F(I,NP,2)=0.0
DO 22 J=1,NVTEX
DBV=YP(J)-YP(J)
DX=XC(I)-XC(J)
DY=YC(I)-YC(J)
R=DX*DX+DY*DY
DO 22 K=1,2
L=3-K
DA=DEL(I,L)
IF(K.EQ.2 .AND. DA.GT.PI2)DA=DA+PI
22 F(I,NP,K)=F(I,NP,K)+(DY*DCOS(DA)-DX*DSIN(DA))/R*DBV
21 CONTINUE

C KUTTA CONDITION
C
DO 30 J=1,NP
IF(J.NE.1)GO TO 31
F(NP,J,1)=F(NP1,1,2)
GO TO 30
31 IF(J.NE.NP1)GO TO 32
F(NP,J,1)=F(1,NP1,2)
GO TO 30
32 F(NP,J,1)=F(NP1,J,2)-F(1,J,2)
30 CONTINUE

C RIGHT HAND SIDE ELEMENTS
C
40 DO 33 J=1,NP1
33 H(J)=-2.0*PI*DCOS(ALFA-DEL(J,2))
IF(ILIFT.EQ.0)GO TO 45
D1=DEL(J,1)
IF(D1.GT.PI2 .OR. D1.LT.(-PI2))D1=D1+PI
D2=DEL(NP,1)
IF(D2.GT.PI2 .OR. D2.LT.(-PI2))D2=D2+PI
H(NP)=2.0*PI*(DCOS(ALFA-D1)-DCOS(ALFA-D2))
C HAVE NOW CONSTRUCTED A MATRIX SYSTEM F*SOR=H WHICH MUST BE
C SOLVED FOR SOR. NB! SOR(NP)=VORTEX STRENGTH, GAMMA
C
45 IF(ILIFT.EQ.0)NP=NP1
FSCALE=F(1,1,1,1)
DO 47 I=1,NP
DO 49 J=1,NP
48 F(I,J,1)=F(I,J,1)/FSCALE
47 H(I)=H(I)/FSCALE
DO 41 K=2,NP
300 FORMAT(1P9E13.3)
301 FORMAT(1H1)
40 L=K-1
DO 41 I=K,NP
42 F(I,J,1)=F(L,J,1)-R*F(I,J,1)
41 H(I)=H(L)-R*H(I)
SOR(NP)=H(NP)/F(NP,NP,1)
DO 43 I=2,NP
K=NP+1-I
K1=K+1
SUM=0.0
DO 44 J=K1,NP
44 SUM=SUM+SOR(J)*F(K,J,1)
43 SOR(K)=(H(K)-SUM)/F(K,K,1)
IF(ILIFT. NE. 0) CL = 2.0 * SOR(NP) * SUMB

CALCULATE TANGENTIAL VELOCITIES AT THE CONTROL POINTS

N1 = NP
IF(ILIFT. NE. 0) N1 = N1 - 1
SUMKC = 0.0
SMAX = 0.0
DO 50 I = 1, N1
  DX = XP(I+1) - XP(I)
  DY = YP(I+1) - YP(I)
  DS = DSORT(DX*DX + DY*DY)
  SUM = 0.0
  IF(SOR(I) .GT. SMAX) SMAX = SOR(I)
  SUMKC = SUMKC + SOR(I) * DS
  DO 51 J = 1, NP
    SUM = SUM + SOR(J) * GAMMA(I, J, 2)
    DA = DEL(I, 1)
    IF(DEL(I, 1) .GT. PI2) DA = DEL(I, 1) - PI
    UT(I) = DCOS(ALFA - DA) + SUM / 2.0 / PI
    UX(I) = UT(I) * DCOS(DA)
    51 SUM = SUM + SOR(J) * GAMMA(I, J, 2)
    DA = DEL(I, 1)
    IF(DEL(I, 1) .GT. PI2) DA = DEL(I, 1) - PI
    UT(I) = DCOS(ALFA - DA) + SUM / 2.0 / PI
    UX(I) = UT(I) * DCOS(DA)
  50 SUMKC = SUMKC / SMAX

OUTPUT

CALF = COS(ALFA)
ALFA = ALFA / PI * 180.0
WRITE(6, 100) ALFA
WRITE(6, 4) (TEXT(J), J = 1, 20)
WRITE(6, 102)
DO 91 I = 1, NP
  WRITE(6, 101) XP(I), YP(I)
  IF(ILIFT. NE. 0) WRITE(6, 107)
  WRITE(6, 103) CM = 0.0
  CLI = 0.0
  DO 90 J = 1, NP1
    DEL(J, 1) = DEL(J, 1) * 180.0 / PI
    DEL(J, 2) = DEL(J, 2) * 180.0 / PI
    CP(J) = 1.0 - UT(J)**2
  90 WRITE(6, 104) XC(J), YC(J), DEL(J, 1), DEL(J, 2), SOR(J), UT(J), UX(J), UY(J), CP(J)
  CLI = 0.0
  CM = 0.0
  CALL FORCE(CLI, CM, CP, XC, XP, NP1)
  IF(DABS(SUMKC) .GT. 1.0D-2) WRITE(6, 108)
  WRITE(6, 109) SUM
  IF(ILIFT. EQ. 0) GO TO 999
  CIRC = SOR(NP) * SUMB
  WRITE(6, 105) CIRC, CL
  CLI = CLI * CALF
  WRITE(6, 110) CLI, CM
999 FORMAT(1HI//’5X,’CALCULATION OF POTENTIAL FLOW’,
’AROUND AN AIRFOIL SECTION’//
’5X,’ANGLE OF ATTACK IS: ’,F6.2//)
100 FORMAT(2F10.6)
101 FORMAT}//’5X,’NORMALIZED INPUT COORDINATE TABLE:’//’5X,’X=’,
’9X,’Y=’//)
SUBROUTINE FORCE(CL,CM,C,XC,XP,NP)
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
DIMENSION C(69),XC(69),XP(70)
DO 2 J=1,NP
2 IF(XP(J+1).GT.XP(J)) GO TO 3
JCR=J+1
DO 4 J=2,JCR1
M=JCR+J
N=JCR-J+1
DCU1=(C(M)-C(M-1))/(XC(M)-XC(M-1))
DCU2=(C(M+1)-C(M))/(XC(M+1)-XC(M))
DCL1=(C(N)-C(N+1))/(XC(N)-XC(N+1))
DCL2=(C(N-1)-C(N))/(XC(N-1)-XC(N))
IF(J.NE.2) GO TO 5
DFL=C(N+1)*XP(N+1)-XP(N+2))
DFM=5*C(N+1)*(XP(N+1)+XP(N+2))-5)*XP(N+1)-XP(N+2))
DCL1=5*C(N+1)*(*XP(N+1)+XP(N+2)-5)*XP(N+1)XP(N+2))
DCL2=5*C(N+1)*(*XP(N+1)+XP(N+2)-5)*XP(N+1)XP(N+2))
DCLM=C(M+1)*XP(M)+XP(M-1)
DUM=5*C(M+1)*(*XP(M)+XP(M-1)-5)*XP(M)XP(M-1))
DCLM=5*C(M+1)*(*XP(M)+XP(M-1)-5)*XP(M)XP(M-1))
DCL1=5*C(M+1)*(*XP(M)+XP(M-1)-5)*XP(M)XP(M-1))
DCL2=5*C(M+1)*(*XP(M)+XP(M-1)-5)*XP(M)XP(M-1))
CL=CL+DFL-DFM
CM=CM-DCLM+DCLM
5 IF(J.NE.JCR1) GO TO 6
DFL=C(1)*XP(1)+XP(2))
DFM=5*C(1)*(*XP(1)+XP(2)-5)*XP(1)+XP(2))
DCL1=5*C(1)*(*XP(1)+XP(2)-5)*XP(1)+XP(2))
DCL2=5*C(1)*(*XP(1)+XP(2)-5)*XP(1)+XP(2))
DCLM=C(NP-1)*XP(NP-1)
DUM=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
DCLM=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
DCL1=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
DCL2=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
CL=CL+DFL-DFM
CM=CM-DCLM+DCLM
6 DFU=C(M+1)*XP(M)-XP(M+2))
DCL1=(XP(M)+XP(M+1)-5)*XP(M)+XP(M+1))
DCL2=(XP(M)+XP(M+1)-5)*XP(M)+XP(M+1))
DCLM=C(NP-1)*XP(NP-1)
DUM=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
DCLM=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
DCL1=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
DCL2=5*C(NP-1)*(*XP(NP)+XP(NP-1)-5)*XP(NP)XP(NP-1))
CL=CL+DFL-DFM
CM=CM-DCLM+DCLM
DMU = C(M,1)*(XC(M)**2-XP(M)**2-(XC(M)-XP(M))/2.)/2.+
+DCU1*(XC(M)**2*(XC(M)/3.-XC(M-1)/2.)-
-XP(M)*(XC(M)/2.-XC(M-1))/4.-XP(M)**2*(XP(M)/3.-XC(M-1)/2.)+
+XP(M)**2*(XP(M)/2.-XC(M-1))/4.)+
+C(M)**(XP(M+1)**2-XC(M)**2-(XP(M+1)-XC(M))/2.)/2.+
+DCU2*(XP(M+1)**2*(XP(M+1)/3.-XC(M)/2.)-
-XP(M+1)**(XP(M+1)/2.-XC(M))/4.+XC(M)**3/6.-XC(M)**2/8.)
-ML=C(N+1)*(XC(N)**2-XP(N+1)**2-(XC(N)-XP(N+1))/2.)/2.+
+DCL1*(XC(N)**2*(XC(N)/3.-XC(N+1)/2.)-
-XP(N)*(XC(N)/2.-XC(N+1))/4.-XP(N)**2*(XP(N+1)/3.-XC(N+1)/2.)+
+XP(N)**2*(XP(N)/2.-XC(N))/4.+XC(N)**3/6.-XC(N)**2/8.)
Fig. 3.1.1 The circular cylinder approximated using 8 and 32 panels.
Fig. 3.1.2 The symmetric NACA 0012 profile.
Fig. 3.1.3 Computed and conformal mapping solution (●) for NACA 0012.
Fig. 3.2.1 The NACA 1408 airfoil section.
Fig. 3.2.2 Calculated (●) lift characteristics using a single vortex.
Fig. 3.2.3 Velocity distribution using a single vortex.
Fig. 3.2.4 Computed (○) lift characteristics.
Fig. 3.2.6 Calculated (○) lift characteristics for NACA 1408.
Fig. 3.2.7 Computed velocity distribution using equation (3.2.3).
Fig. 3.2.8 Pressure distribution using equation (3.2.4).
Fig. 3.2.9 Calculated pressure distribution for NACA 1408 at $\alpha = 4^0$. 
Fig. 4.1 Calculated (○) and measured lift characteristics for NACA 0012.
Fig. 4.2 The NACA 4412 airfoil section.
(Not differences in scales)
Fig. 4.3 Calculated (○) and measured lift characteristics for NACA 4412.
Fig. 4.4 Calculated pressure distribution at zero angle of attack.
Fig. 4.5 Computed and measured distribution at $\alpha_{\text{geom}} = 2^\circ$ and $\alpha_{\text{eff}} = 0.44$. 
Fig. 4.6 Computed and measured pressure field at $\alpha_{\text{geom}} = 16^\circ$ and $\alpha_{\text{eff}} = 11.32^\circ$. 
Fig. 4.7 Computed and measured (○) pressure distribution for the swept wing.
Fig. 4.8 The Korn-Garabedian airfoil section.
Fig. 4.9 Computed and measured (○) pressure coefficients.
LIEBECK #33  53 PANELS

Fig. 4.10 High lift airfoil section.
LIEBECK #33
ALFA=10DEG

Fig. 4.11 Calculated and measured (■) pressure distribution.
Fig. 4.12 Comparison between computed and analytical solution (▲).