ON CYLINDRICAL SOLITARY WAVES

by

Henry Power Meneses and Allen T. Chwang

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Iowa Institute of Hydraulic Research
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**Abstract (Limit: 200 words)**
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The main objective of this study is to investigate analytically the propagation and focusing of a cylindrical solitary wave in waters of constant depth. In order to accomplish this goal, we applied an inner-outer expansion technique to the cylindrical Boussinesq equations. The results are in good agreement with the numerical solutions found in the literature. The behavior of the cylindrical solitary wave in the outer region is investigated by obtaining an approximate solution of the corresponding cylindrical Korteweg-de Vries equation.

Of particular interest is the analysis of the three basic laws of conservation for a physical system. These three conservation laws, for the cylindrical Korteweg-de Vries equation have been discussed in detail.

**Descriptors**
- Water Waves, Solitary Waves, Tsunamis
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\( N \) \_\_n \quad n\text{th order of } N \text{ in the perturbation expansion}

\( N* \) \quad \text{Wave amplitude in the inner region}

\( \tilde{N} \) \quad N*/h

\( n \) \quad \text{Integer}

\( O(\varepsilon) \) \quad \text{Order of } \varepsilon

\( Q_n \) \quad \text{Integrand of the integral invariant}

\( r* \) \quad \text{Radial direction}

\( r_{\text{max}} \) \quad \text{Position of the maximum of } \eta

\( r \) \quad \text{Dimensionless radial direction}

\( r_0 \) \quad \text{Initial location of the wave}

\( s \) \quad -\zeta

\( t* \) \quad \text{Time}

\( t_{\text{max}} \) \quad \text{Time of the maximum of } \eta

\( t \) \quad \text{Dimensionless time}

\( t_0 \) \quad \text{Initial time}

\( t_1 \) \quad \text{Dimensionless time when the wave reaches the origin}

\( t_2 \) \quad \text{Dimensionless time when the wave leaves the origin}

\( \tilde{t} \) \quad t*(g/h)^{1/2}

\( U \) \quad \text{Depth-averaged radial velocity in the inner region}

\( U_n \) \quad n\text{th order of } U \text{ in the perturbation expansion}

\( u* \) \quad \text{Depth-averaged radial velocity}

\( u \) \quad \text{Dimensionless depth-averaged radial velocity}

\( u_n \) \quad n\text{th order of } u \text{ in the perturbation expansion}

\( u_r \) \quad \text{Dimensionless radial velocity}

\( u_z \) \quad \text{Dimensionless vertical velocity}

\( u_o \) \quad \text{Zeroth-order incident velocity}
\( u_{o,\tau} \)
Zeroth-order reflected velocity

\( V^* \)
Volume

\( \nu \)
\( \eta(\tau/\tau_o)^{1/2} \)

\( W \)
Vertical velocity in the inner region

\( w(v) \)
Width function

\( X \)
\( \sqrt{\frac{3\epsilon}{2\mu}} [(x-x_o) - (1 + \frac{\epsilon}{2})t] \)

\( x^* \)
Horizontal coordinate

\( x \)
Dimensionless horizontal coordinate

\( x_o \)
Initial location of the wave

\( x_1 \)
Constant

\( x_m \)
Position of the maximum of \( \eta \) at \( t = 2x_o/(1 + \epsilon/2) \)

\( z^* \)
Vertical axis

\( z \)
Dimensionless vertical axis

\( \alpha \)
Outer variable for incoming wave

\( \beta \)
Outer variable for outgoing wave

\( \nu \)
\( \frac{1}{2} (\tau^{1/2} - \tau_o^{1/2}) \)

\( \delta_1 \)
\( \frac{\mu}{2\sqrt{3}} \tanh \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) - 1 \)

\( \delta_2 \)
\( \frac{\mu}{2\sqrt{5}} \tanh \left( \frac{\sqrt{5} \epsilon}{2\mu} \alpha \right) + 1 \)

\( \epsilon \)
\( a_o/h \)

\( \xi \)
\( r + t \)

\( \eta^* \)
Wave amplitude measured from the undisturbed water surface

\( \eta \)
Dimensionless wave amplitude

\( \eta_{max} \)
\( a(\tau) \)

\( \eta_n \)
nth order of \( \eta \) in the perturbation expansion

\( \eta_{o_1} \)
Zeroth-order incident wave
\( \eta_0 \)  Zeroth-order reflected wave

\( \theta \)  \( \lim_{x \to 0} \)

\( \lambda \)  Wave length

\( \mu \)  \( k_0 \)

\( \mu_n \)  nth order of \( \mu \) in the perturbation expansion

\( \xi \)  Inner horizontal coordinate

\( \pi \)  3.14159...

\( \rho \)  Mass density of fluid

\( \sigma \)  \( r - t \)

\( \sigma_0 \)  \( r_0 - t_0 \)

\( \tau \)  \( ct \)

\( \tau_0 \)  Initial value of \( \tau \)

\( \phi^* \)  Velocity potential

\( \phi \)  Dimensionless velocity potential

\( \phi_n \)  nth order of \( \phi \) in the Taylor series

\( \psi \)  \( \frac{\xi}{\mu_0} \frac{\mu(x-ct)}{(\sigma - \sigma_0) - \nu} \)

\( \Omega \)  \( \frac{
\Omega}{w(\nu)} \)

\( \Delta t \)  Phase lag

\( \Delta x \)  Phase shift
CHAPTER I
INTRODUCTION

Many surface wave problems have been studied by using the linearized approximation method. One of the main advantages gained is that superposition of several solutions is possible. However, there are circumstances where the linear theory ceases to hold, for a variety of reasons, and non-linear phenomena will affect the wave propagation. Some effects are drastic, like breaking waves, and others are mild so that significant modifications may be detected only after a long time, as in the case of energy exchange through non-linear interactions in the wave spectrum.

It is known that in shallow water, the linearized approximation is useful only if the following two length ratios are very small,

\[ \varepsilon = \frac{a_o}{h} \ll 1 \text{ and } \mu = \frac{k_o}{h} \ll 1, \]

where \( h \) is the water depth, \( a_o \) the maximum wave amplitude, and \( k_o \) the wave number, \( k_o = \frac{2\pi}{\lambda} \) with \( \lambda \) being the wavelength. The first restriction is very severe for typical amplitudes. Hence, the non-linear theory is important to many practical problems. The ratio between these two dimensionless parameters gives the well known Stoker-Ursell number, \( K = \frac{\varepsilon}{\mu^2} \). This new parameter \( K \) plays an
important role in the development of nonlinear water-wave theories as is the case in the Stokes theory of finite amplitude waves, in which the Stoker-Ursell parameters must be small.

When both $\varepsilon$ and $\mu^2$ are small, but are of the same order, that is when $K$ is of the order of $O(1)$, Boussinesq equations will be the correct governing equations. These equations contain non-linear as well as dispersive terms, and they are valid for long waves in water of constant depth moving in both directions (incoming and outgoing). The Korteweg-de Vries (KdV) equation is derived from the Boussinesq equations by following the wave in one direction only. The one-dimensional KdV equation (in its dimensionless form) for waves propagating in the positive $x$ direction is given by (see equation (2.31))

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2} \varepsilon \eta \frac{\partial \eta}{\partial x} + \frac{\mu^2}{6} \frac{\partial^3 \eta}{\partial x^3} = 0(\mu^2 \varepsilon, \mu^n),$$

(1-1)

where $\eta(x,t)$ is the wave amplitude measured from the undisturbed water surface and $\eta_{\text{max}} = 1$.

Although the above equation was first formulated for water waves, it is well known that this equation governs the propagation of many one-dimensional, weakly non-linear dispersive waves in different physical systems such as acoustic waves in lattices, and ion-acoustic waves in a collisionless plasma, etc.

Great interest in the propagation of one-dimensional long waves has been generated by the exact solutions of the KdV equation found by Miura et al. (1968). As a natural extension, a new interest in the propagation of three-dimensional long waves has been aroused in recent years.
In this study we are concerned with the propagation and reflection of a cylindrical solitary wave in water of constant depth.

1.1 Literature Survey

Since Hershkowitz and Romesser (1974) observed experimentally that a cylindrical ion-acoustic solitary wave propagating toward the origin travels somewhat faster than the corresponding one-dimensional solitary wave, and that the wave width multiplied by the square root of the maximum wave amplitude is approximately constant even though both the amplitude and width are functions of time, a great deal of interest in the propagation of cylindrical solitary waves has been generated. Maxon and Viecelli (1974) have derived a modified KdV equation for axisymmetric cylindrical waves propagating in a collisionless plasma. They solved this modified KdV equation numerically and found that the wave amplitude grows initially like $r^{-1/2}$ as $r$ decreases; and the product of the square root of the wave amplitude and the wave width remains approximately constant. Ogino and Takeda (1976) pointed out that Maxon and Viecelli's numerical solution is not valid when the wave approaches the origin due to the reflection during the focusing process. They studied the propagation of a cylindrical ion-acoustic solitary wave based on a numerical integration of the cylindrical Boussinesq equations, and they obtained an approximate solution for the early stage of propagation, which behaves like Maxon and Viecelli's solution.

Chwang and Wu (1976) investigated the propagation and focusing of a cylindrical solitary wave in water of constant and variable depth. They solved these problems by integrating the cylindrical Boussinesq
equations, adopting a finite-difference scheme. For the case of a wave propagating in water of constant depth, their solution consists of a main positive wave whose maximum amplitude initially grows like $r^{-1/2}$ as $r$ decreases and a small but persistent negative wave which follows the main wave. When the crest of the incoming wave reaches the origin at $r = 0$, it remains there for a period of time, during which the wave reaches its maximum amplitude, afterwards the crest of the wave begins to move away from the origin as the wave changes its direction.

Cumberbatch (1978) found an approximate similarity solution for the cylindrical KdV equation, which resembles the main part of the wave. The maximum amplitude of this solution grows like $r^{-2/3}$ as $r$ decreases, while the width decreases like $r^{1/3}$. Another similar solution of the cylindrical KdV equation was obtained by Miles (1978). His solution indicates that the wave amplitude grows like $r^{-2/3}$ as $r$ decreases and the main wave is followed by a train of always positive or negative waves whose shape is approximately given by the square of an Airy function.

Ko and Kuehl (1979) found a small-time perturbation solution of the cylindrical KdV equation. The major conclusion of their work is the existence of a very small continual transfer of energy from the main wave to the trailing structure. Johnson (1979) proposed a change of variables in order to transform the cylindrical KdV equation into the Kadomtsev-Petviashvili equation so that the integral equation solution of the latter, based on the inverse scattering transform, may be applied.
From this literature survey we conclude that there is a discrepancy between the existent solutions in this field which needs to be clarified by a complete analytic solution.

1.2 Outline of the present investigation

The objective of the present investigation is to develop an analytical solution for the propagation and focusing of a cylindrical solitary wave in water of constant depth based on the inner and outer expansions of the cylindrical Boussinesq equations.

In Chapter II we present the equations which govern the propagation of three-dimensional long waves in water of constant depth based on the assumption that the fluid is incompressible and inviscid and the flow irrotational. The cylindrical Boussinesq equations are derived. These equations contain non-linear and dispersive terms and govern the propagation of cylindrical long waves moving in both directions. Then the cylindrical KdV equation is obtained from the former system by specifying that the wave moves in one direction only.

Of particular interest is the analysis of the three basic laws of conservation for a physical system: conservation of mass, conservation of momentum, and conservation of energy. In Chapter III we discuss these three conservation laws for the cylindrical KdV equation.

In Chapter IV, the behavior of the cylindrical solitary wave for the incoming and out-going propagation in water of constant depth in the outer region is investigated by obtaining an approximate solution of the corresponding cylindrical KdV equation.
When the wave approaches the origin the cylindrical KdV equation is no longer valid due to the reflection during the focusing process. At that time the cylindrical Boussinesq equations should be used. An analytical solution for this focusing process is presented in Chapter V section 5.2. In section 5.1, we first solve the problem of reflection of a planar solitary wave at a vertical wall in order to develop the necessary mathematical technique which is used in section 5.2.

In Chapter VI we present the major conclusions drawn from this study.
CHAPTER II
CYLINDRICAL BOUSSINESQ AND KORTEWEG-de VRIES EQUATIONS

In this Chapter we shall present the equations which govern the propagation of three-dimensional long waves in water of constant depth. The fluid is assumed to be incompressible and inviscid and the motion irrotational; hence the velocity field has a scalar potential \( \phi \). The vertical \( z \) axis is measured from the still water level. The displacement of the free surface from the still water level is \( \eta^*(r^*, t^*) \) and the solid bottom is at \( z^* = -h \). The fluid is supposed to be unbounded in the radial direction (\( 0 \leq r^* < \infty \), see Figure 1).

We introduce the following dimensionless variables:

\[
\begin{align*}
    r &= k_o r^*, \\
    z &= z^*/h, \\
    t &= k_o \sqrt{gh} t^* \\
    \eta &= \eta^*/a_o, \\
    \phi &= \phi^*/(k_o a_o \sqrt{gh}),
\end{align*}
\]

where all the variables are non-dimensionalized by using the characteristic wave number \( k_o = 2\pi/\lambda_o \) (\( \lambda_o \) being the wavelength) and the characteristic wave amplitude \( a_o \). The above dimensionless variables imply the following normalization for the velocity components:

\[
    u_r = \frac{\partial \phi}{\partial r} = \frac{h}{a_o \sqrt{gh}} \frac{\partial \phi^*}{\partial r^*},
\]
\[ u_z = \frac{\partial \phi}{\partial z} = \frac{k_o h^2}{a_o \sqrt{gh} \frac{\partial^2 \phi}{\partial z^2}}. \]

The velocity potential \( \phi \) satisfies the dimensionless Laplace equation

\[ \mu^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \right] + \frac{\partial^2 \phi}{\partial z^2} = 0 \]

\((-1 < z < \varepsilon \eta), \quad (2-1a)\]

where \( \varepsilon = a_o / h \) and \( \mu = k_o h. \) \( (2-1b) \)

The boundary conditions are:

(i) Kinematic boundary condition at the free surface

\[ \mu^2 \left[ \frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \phi}{\partial r} \frac{\partial \eta}{\partial r} \right] = \frac{\partial \phi}{\partial z} \]

at \( z = \varepsilon \eta; \quad (2-2) \)

(ii) Dynamic boundary condition at the free surface

\[ \mu^2 \left( \frac{\partial \phi}{\partial t} + \eta \right) + \frac{1}{2} \varepsilon \left[ \mu^2 \left( \frac{\partial \phi}{\partial r} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0 \text{ at } z = \varepsilon \eta; \quad (2-3) \]

(iii) No normal flow through the bottom boundary

\[ \frac{\partial \phi}{\partial z} = 0 \text{ at } z = -1. \quad (2-4) \]
Since \( \phi \) is analytic we may expand it in a Taylor series as

\[
\phi(r,z,t) = \sum_{n=0}^{\infty} (z+1)^n \phi_n(r,t).
\] (2-5)

Using \( \nabla \) to denote the horizontal gradient in polar coordinate

\[
\left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta} \right), \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r},
\]

we have the following derivatives:

\[
\nabla \phi = \sum_{n=0}^{\infty} (z+1)^n \nabla \phi_n, \quad (2-6a)
\]

\[
\nabla^2 \phi = \sum_{n=0}^{\infty} (z+1)^n \nabla^2 \phi_n, \quad (2-6b)
\]

\[
\frac{\partial \phi}{\partial z} = \sum_{n=0}^{\infty} (z+1)^n (n+1) \phi_{n+1}, \quad (2-6c)
\]

\[
\frac{\partial^2 \phi}{\partial z^2} = \sum_{n=0}^{\infty} (z+1)^n (n+2) (n+1) \phi_{n+2}. \quad (2-6d)
\]

Substitution (2-6) into the Laplace equation (2-1), we obtain

\[
\mu^2 \nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} = \sum_{n=0}^{\infty} (z+1)^n \left[ \mu^2 \nabla^2 \phi_n + (n+2) (n+1) \phi_{n+2} \right] = 0. \quad (2-7)
\]

Since \( z \) is arbitrary within \((-1,\varepsilon n)\) the coefficient of each power of \((z+1)\) must vanish. Hence,

\[
\phi_{n+2} = \frac{-\mu^2 \nabla^2 \phi_n}{(n+2)(n+1)}. \quad (2-8)
\]

From the bottom boundary condition equation (2-4), we obtain

\[
\phi_1 \equiv 0.
\]
Therefore, from equation (2-8) we find that all \( \phi_n \)'s with odd \( n \) vanish,

\[
\phi_1 = \phi_3 = \phi_5 = \ldots = 0.
\]  

(2-9)

Thus, with an error of \( O(\mu^6) \), the potential \( \phi \) becomes

\[
\phi = \phi_0 - \frac{\mu^2}{2} (z+1)^2 \nabla^2 \phi_0 + \frac{\mu^4}{4!} (z+1)^4 \nabla^2 \nabla^2 \phi_0 + O(\mu^6),
\]  

(2-10)

where \( \phi_0 \) is of the order of \( O(1) \).

Substituting equation (2-10) into the free surface boundary conditions yields

\[
\mu^2 \{ \eta_\tau + \nabla (1 + \epsilon \eta) \cdot [\nabla \phi_0 - \frac{\mu^2}{2} (1 + \epsilon \eta)^2
\]

\[
\nabla (\nabla \phi_0) \} = -\mu^2 (1 + \epsilon \eta) \nabla^2 \phi_0 + \frac{\mu^4}{6} (1 + \epsilon \eta)^3 \nabla^2 \nabla^2 \phi_0
\]

+ \( O(\mu^6) \),  

(2-11)

and

\[
\mu^2 \{ \phi_0 \cdot \nabla - \frac{\mu^2}{2} (1 + \epsilon \eta)^2 \nabla^2 \phi_0 \cdot \eta_\tau + \eta \} + \frac{\epsilon}{2} \mu^2 [\nabla \phi_0]^2
\]

\[
- \mu^2 (1 + \epsilon \eta)^2 \nabla \phi_0 \cdot \nabla (\nabla \phi_0) + \frac{1}{2} \epsilon \mu^4 (1 + \epsilon \eta)^2 (\nabla^2 \phi_0)^2
\]

= \( O(\mu^6) \),  

(2-12)
where the subscript $t$ denotes the derivative with respect to time.

Equations (2-11) and (2-12) may be simplified as

$$
\eta_t + \nabla \cdot [(1 + \epsilon \eta) \nabla \phi_0] - \frac{\mu^2}{6} \nabla^4 \phi_0 = 0(\epsilon \mu^2, \mu^4) \tag{2-13}
$$

and

$$
\phi_{0t} + \eta + \frac{\epsilon}{2} (\nabla \phi_0)^2 - \frac{\mu^2}{2} \nabla^2 \phi_0 = 0(\epsilon \mu^2, \mu^4). \tag{2-14}
$$

The depth-averaged radial velocity $u$ is defined as

$$
u = \frac{1}{1 + \epsilon \eta} \int_{-1}^{\eta} \frac{\partial \phi}{\partial r} \, dz. \tag{2-15}
$$

Substituting (2-10) into (2-15), we obtain

$$
u = \frac{\partial \phi_0}{\partial r} - \frac{\mu^2}{6} \left[ \nabla^2 \left( \frac{\partial \phi_0}{\partial r} \right) - \frac{1}{r^2} \frac{\partial \phi_0}{\partial r} \right] + 0(\epsilon \mu^2, \mu^4). \tag{2-16}
$$

Using equation (2-16), we can simplify (2-13) to

$$
\eta_t + \frac{1}{r} [(1 + \epsilon \eta) ru]_r = 0(\epsilon \mu^2, \mu^4). \tag{2-17}
$$

Taking the derivative of equation (2-14) with respect to $r$ and making use of (2-16), we obtain

$$
u_t + \epsilon \mu u_r + \eta_r - \frac{\mu^2}{3} \left[ \frac{1}{r} (ru)_r \right]_{rr} = 0(\epsilon \mu^2, \mu^4). \tag{2-18}
$$
We note that $c$ and $\mu^2$ are assumed to be of the same order.

Equations (2-17) and (2-18) contain non-linear and dispersive terms, and they govern the propagation of axisymmetric long waves in water of constant depth moving in both directions (increasing $r$ direction and decreasing $r$ direction). The two-dimensional version of the above system of equations is found to be

$$\eta_t + [(1 + \epsilon \eta)u]_x = 0(\epsilon \mu^2, \mu^4), \tag{2-19}$$

and

$$u_t + \epsilon uu_x + \eta_x - \frac{\mu^2}{3} u_{xx} = 0(\epsilon \mu^2, \mu^4). \tag{2-20}$$

This system of equations is the well known Boussinesq equations which govern the propagation of one-dimensional long waves.

The cylindrical KdV equation is derived from equations (2-17) and (2-18) by specifying the propagating direction of the waves. For an out-going wave we assume that the wave profile will change very slowly if we follow the wave with a phase velocity $c = 1$ which corresponds to a dimensional phase velocity of $\sqrt{g \mu}$. Therefore, we shall define a new pair of variables $\sigma$ and $\tau$ by

$$\sigma = r - t, \quad \tau = \epsilon t. \tag{2-21}$$

In terms of these new variables the derivatives $\frac{3}{\partial \tau}$ and $\frac{3}{\partial t}$ become
(a) There is a positive pressure region in front of the jet and a negative pressure region in its wake. On the jet-exit plane the dividing zero pressure contour is swept farther downwind for smaller $K$ values or higher wind velocities. The absolute value of the pressure coefficient, $p/(1/2 \rho U_a^2)$, falls to below 0.1 within about three orifice diameters beyond the orifice. At some distance away from the orifice, the pressure coefficient decreases in absolute value with increasing $K$ values. The distribution of the pressure coefficient along the centerline has a trend in front of the jet similar to that in potential flow around a two-dimensional circular cylinder. In the wake region the pressure is lower than in the case of uniform flow past a circular cylinder with turbulent boundary layer at comparable Reynolds number. The local drag coefficient calculated from the $x$-component of the pressure force on the jet-exit plane varies from 2.92 for $K = 2.26$ to 0.8 for $K = 8.77$.

(b) The approximate similarity profiles given by (6-1) and (6-6) were obtained for the vertical and total velocities, respectively, along horizontal lines in the plane of symmetry. The similarity profiles appear to depend on the polar angle $\phi$ and $K$ value through its effects on turbulence production and distribution. Similarity profiles for smaller $K$ values thus have smaller lateral extent in the centerplane, and vice versa.

(c) The total centerline velocity of jet decays much more rapidly along the trajectory of a jet in a cross-flow than along a free jet. Theoretical predictions indicate the variation of the mean velocity
over the jet cross-sections. The value for $\beta$ determined from velocity data, while fixing $\alpha$ at its free-jet value, varies from 0.05 to 0.24 for $K = 8.77$ to 4.24. The velocity is approximately an exponential function of the normalized jet elevation, $y/\Delta K$. For small $y/\Delta K$, the first two terms of the series expansion of the exponential function give a good approximation to the experimental data.

(d) Use of $\beta$ values determined from the velocity data yielded theoretical trajectories which agree reasonably well with experimental results in the near-field zone; $C_D$ values determined therefrom are almost negligible. Thus, the drag force appears to be important only in the potential core zone. The jet trajectory in this zone also plots as a straight line on semi-logarithmic paper, as suggested by an approximation of the complete equation, (6-13).

(e) The range of applicability for the curvilinear zone solution is rather limited, perhaps between $y/\Delta K = 0$ and $y/\Delta K = 2$, while the $1/3$-power relation apply over a very large range of the jet trajectory, from approximately $y/\Delta K = 1.5$ up to $y/\Delta K = 10$, the limit of observed trajectories, the momentum length $\Delta K$ being the sole scaling factor there.

(f) The horizontal half-width of the jet seems to increase exponentially with $y/\Delta K$.

(g) The control-volume experiments revealed that over the potential-core region, the fluid entrainment rate is relatively small and the drag force exerted on the jet by the pressure distribution around it produces most of the momentum exchange between the jet and the cross-flow. In this region, $C_D$ has a value in excess of 3.0, whereas the
\[ \frac{\partial \eta}{\partial t} = \frac{3}{\partial \sigma}, \quad (2-22a) \]

\[ \frac{3}{\partial t} = \frac{3}{\partial \sigma} + \varepsilon \frac{3}{\partial t}. \quad (2-22b) \]

Substituting (2-21) and (2-22) into equations (2-17) and (2-18), we obtain

\[ \varepsilon \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial \sigma} + \frac{\varepsilon}{\partial \sigma} \left[ u + \frac{(\varepsilon \sigma + \tau)}{\varepsilon} \frac{\partial u}{\partial \sigma} + \right. \]

\[ \varepsilon \eta u + (\varepsilon \sigma + \tau) \frac{\partial (\eta u)}{\partial \sigma} ] = 0(\mu^4), \quad (2-23) \]

and

\[ \varepsilon \frac{\partial u}{\partial t} - \frac{\partial u}{\partial \sigma} - \varepsilon u \frac{\partial u}{\partial \sigma} + \frac{\partial \eta}{\partial \sigma} + \frac{\mu^2}{3} \left\{ \frac{\varepsilon}{\partial \sigma} \left[ \frac{(\varepsilon \sigma + \tau)}{\varepsilon} u \right] \right\}_{\sigma \sigma} \]

\[ - \varepsilon \frac{\mu^2}{3} \left\{ \frac{\varepsilon}{\partial \sigma} \left[ \frac{(\varepsilon \sigma + \tau)}{\varepsilon} u \right] \right\}_{\sigma \tau} = 0(\mu^4). \quad (2-24) \]

The last term in equation (2-24) is of the order of \(0(\mu^4)\) since \(\varepsilon\) and \(\mu^2\) are of the same order for the Boussinesq waves. Hence, this equation can be written as

\[ \varepsilon \frac{\partial u}{\partial t} - \frac{\partial u}{\partial \sigma} - \varepsilon u \frac{\partial u}{\partial \sigma} + \frac{\partial \eta}{\partial \sigma} + \frac{\mu^2}{3} \left\{ \frac{\varepsilon}{\partial \sigma} \left[ \frac{(\varepsilon \sigma + \tau)}{\varepsilon} u \right] \right\}_{\sigma \sigma} = 0(\mu^4). \quad (2-25) \]

To the lowest order, neglecting terms of the order \(\varepsilon\) and \(\mu^2\) in equations (2-23) and (2-25), we have
\[ \frac{\partial \eta}{\partial \sigma} = \frac{\partial u}{\partial \sigma}, \quad (2-26) \]

Integrating the above equation, we obtain

\[ \eta = u + A(\tau), \quad (2-27) \]

where \( A(\tau) \) is an arbitrary function of time. Now, if \( \eta \) and \( u \) approach zero as \( \sigma \) goes to plus or minus infinity for all values of \( \tau \), we find that

\[ A(\tau) \equiv 0. \]

Therefore,

\[ \eta = u + O(\epsilon). \quad (2-28) \]

If we add equations (2-23) and (2-25) and use equation (2-28), we obtain

\[ \frac{\partial \eta}{\partial \tau} + \frac{3}{2} \frac{\partial \eta}{\partial \sigma} + \frac{1}{2} \frac{n}{\tau} + \frac{1}{6} \frac{\mu^2}{\epsilon} \frac{\partial^3 \eta}{\partial \sigma^3} = O(\mu^2). \quad (2-29) \]

In terms of the original dimensionless variables the above equation can be written as

\[ \frac{\partial \eta}{\partial \tau} + \frac{1}{2} \frac{n}{\tau} + (1 + \frac{3}{2} \epsilon n) \frac{\partial n}{\partial r} + \frac{\mu^2}{6} \frac{\partial^3 \eta}{\partial r^3} = O(\mu^4). \quad (2-30) \]
This is the cylindrical KdV equation for waves moving in the positive r direction. This equation was first formulated by Maxon and Viecelli (1974) for an ion-acoustic wave propagating in a collisionless plasma, and it was applied later to long waves propagating in water of constant depth by Miles (1978).

The two-dimensional version of equation (2-30) is the well known KdV equation,

$$\frac{\partial n}{\partial t} + \left(1 + \frac{3}{2} \epsilon n\right) \frac{\partial n}{\partial x} + \frac{\mu^2}{6} \frac{\partial^3 n}{\partial x^3} = 0(\mu^4).$$  \hspace{1cm} (2-31)

Thus, equation (2-30) is the KdV equation plus an additional term $n/2r$.

It was shown by Miura et al. (1968) that the KdV equation has an infinite number of conservation quantities (invariants) of the form

$$I_n = \int_{-\infty}^{\infty} Q_n(x,t) \, dx.$$

The first three invariants correspond to the conservation laws for the mass, momentum and energy. In the next Chapter we shall discuss these three conservation laws for the cylindrical KdV equation.
CHAPTER III
CONSERVATION LAWS FOR THE CYLINDRICAL
KORTEWEG-de VRIES EQUATION

3.1 Conservation of mass

The excess mass of the fluid under a cylindrical solitary wave with respect to the still water level is

\[ M^* = \int_0^\infty \int_0^\infty 2\pi \rho \, r^* \, dr^* \, dz^*. \]

In terms of the dimensionless variables, the above integral can be written as

\[ M^* = \frac{2\pi \rho \rho h}{k^2} \int_0^\infty r \, ndr. \quad (3-1) \]

In the absence of sources or sinks, the mass of the fluid is conserved. The mathematical equivalent of this statement of mass conservation is that the Lagrangian derivative \( \frac{D}{Dt^*} \) of \( M^* \) is equal to zero, that is

\[ \frac{DM^*}{Dt^*} \equiv 0, \quad (3-2a) \]

where

\[ \frac{D}{Dt^*} = k_0 \sqrt{gh} \left\{ \frac{\partial}{\partial t} + \epsilon_\phi \frac{\partial}{\partial r} + \epsilon_{\phi z} \frac{\partial}{\partial z} \right\} \left\{ \frac{\partial}{\partial t} + \epsilon_\phi \frac{\partial}{\partial r} + \epsilon_{\phi z} \frac{\partial}{\partial z} \right\}. \quad (3-2b) \]
Since $M^*$ given by (3-1) is a function of $t$ only, equation (3-2a) reduces to

$$\frac{3}{\delta t} \int_0^{\infty} r_n \, dr = 0. \quad (3-3)$$

Multiplying the cylindrical KdV equation (2-30) by $r$ and integrating over $0 < r < \infty$, we obtain

$$\frac{3}{\delta t} \int_0^{\infty} r_n \, dr + \int_0^{\infty} r \, \frac{3}{\delta r} \, dr + \frac{3}{2} \varepsilon \int_0^{\infty} r \, \frac{3}{\delta r^3} \, dr = 0(\varepsilon^2). \quad (3-4)$$

Assuming that $n$, $n_r$ and $n_{rr}$ vanish at infinity (at least to the order of $o(1/r)$) and are finite at the origin $r = 0$, we have

$$\int_0^{\infty} r \, \frac{3}{\delta r} \, dr = - \int_0^{\infty} n \, dr,$$

$$\int_0^{\infty} r_n \, \frac{3}{\delta r} \, dr = - \frac{1}{2} \int_0^{\infty} n^2 \, dr,$$

$$\int_0^{\infty} r \, \frac{3}{\delta r^3} \, dr = 0.$$

Substituting the above relations into equation (3-4), we have

$$\frac{3}{\delta t} \int_0^{\infty} r_n \, dr = \frac{1}{2} \left[ \int_0^{\infty} n \, dr + \frac{3}{2} \varepsilon \int_0^{\infty} n^2 \, dr \right] + 0(\varepsilon^2). \quad (3-5)$$
Therefore, neglecting terms of the order of \(O(\varepsilon)\), we require that

\[
\int_{0}^{\infty} \eta \, dr = 0. \tag{3-6}
\]

As was found by Lamb (1902) in his linear solution of a cylindrical solitary wave produced by a point source of sound, the cylindrical wave consists of a primary condensation wave followed by a rarefaction wave of lesser amount but lasting for a long time (see figure 2).

From equation (3-5) we note that if only \(\varepsilon^2\) terms are neglected, the mass conservation principle implies that

\[
\int_{0}^{\infty} \eta \, dr = -\frac{3}{2} \varepsilon \int_{0}^{\infty} \eta^2 \, dr. \tag{3-7}
\]

Thus, we can conclude that a single positive jump solution or a train of always positive waves is not a possible solution of the cylindrical KdV equation. Chwang and Wu (1976) analyzed the evolution of a single wave with an initial \(\text{sech}^2\) shape by a numerical integration of the cylindrical Boussinesq equations. Their solution clearly shows a positive main wave followed by a negative wave of small amplitude but long length.

This double-wave phenomenon produced by an initial single hump is characteristic of the three-dimensional wave. It may be attributed to the nature of the geometric distortion in the continuity equation, and not to the non-linearity of the problem, as can be seen from the previous analysis.
3.2 Conservation of momentum

The mass of fluid per unit volume is \( \rho \) and its momentum in the \( r \) direction is \( \frac{\partial \phi}{\partial r} \). The ratio of the momentum in the \( z \) direction to that in the \( r \) direction is of the order of \( O(\varepsilon^{1/2}) \) since the vertical velocity is of the order of \( O(\mu \partial \phi / \partial r) \). The total momentum in the \( r \) direction of the system is given by the following integral over the total volume \( \mathcal{V} \),

\[
m^* = \int_{\mathcal{V}} \frac{\partial \phi}{\partial r} \, dV^*.
\] (3-8)

In terms of our dimensionless variables equation (3-8) can be written as

\[
m^* = \frac{2\pi \varepsilon h \sqrt{gh}}{k_0^2} \int_0^\infty \int_{-1}^{\varepsilon \eta} r \frac{\partial \phi}{\partial r} \, dz \, dr.
\]

Using the definition of the radial average velocity given by (2-15), the above equation becomes

\[
m^* = \frac{2\pi \rho h}{k_0^2} \sqrt{gh} \int_0^\infty \varepsilon ru(1+\varepsilon \eta) \, dr.
\] (3-9)

Substituting the leading order approximation of the cylindrical Boussinesq equations, \( u = \eta + O(\varepsilon) \), into equation (3-9), we obtain

\[
m^* = \frac{2\pi \rho h}{k_0^2} \sqrt{gh} \varepsilon \int_0^\infty \eta \, dr + O(\varepsilon^2).
\] (3-10)

Thus, the equation for the conservation of momentum in the \( r \) direction is
\[
\frac{\partial}{\partial t} \int_0^\infty r n \, dr = 0, \quad (3-11)
\]

after using the same arguments as in the previous section. This equation is the same as equation (3-3). Therefore, the conservation of mass implies the conservation of momentum in the \( r \) direction.

3.3 Conservation of energy

The presence of a progressive wave in a body of water of constant depth makes two contributions to the total energy of the system. One is associated with the kinetic energy of the wave motion, and the other with the potential energy of the deformation of the horizontal surface of the quiescent fluid. Hence, the total energy of the system, the sum of potential and kinetic energies, is

\[
E^* = \int_{V^*} \rho \left[ \frac{1}{2} \left( \frac{\partial \phi^*}{\partial r^*} \right)^2 + g z^* \right] \, dV^* + O(\epsilon). \quad (3-12)
\]

We note that the vertical velocity is of the order of \( O(\epsilon^{1/2}) \). The dimensionless form of the above integral is

\[
E^* = 2\pi \rho \left( \frac{gh^2}{k_0} \right) \left\{ \int_0^\infty \frac{u^2}{2} (1 + \epsilon \eta) \, r \, dr + \int_0^\infty \frac{1}{2} (\epsilon^2 \eta^2 - 1) \, r \, dr + O(\epsilon^3) \right\}. \quad (3-13)
\]

The last term of the second integral in equation (3-13) is the potential energy of the fluid at rest and is unrelated to the presence of the wave. If we use the leading-order approximation of the
Boussinesq equations, \( u = \eta + O(\varepsilon) \), the total energy associated with the wave can be written as

\[
E_w^* = 2\pi \rho \frac{gh^2}{k_0} \left[ \int_0^\infty \varepsilon^2 r \eta^2 \, dr + O(\varepsilon^3) \right]. \tag{3-14}
\]

Substituting the variables \( \sigma \) and \( \tau \) defined by equation (2-21) into equation (3-14) and neglecting terms of \( O(\varepsilon^2) \), we have

\[
E_w^* = 2\pi \rho \frac{gh^2}{k_0} \left\{ \int_{-\infty}^\infty \tau \eta^2 \, d\tau + \varepsilon \left[ \int_{-\infty}^\infty \sigma \eta^2 \, d\tau \right.ight.
\]

\[
+ \left. \int_{-\infty}^\infty \tau \eta^2 \, d\sigma \right\}. \tag{3-15}
\]

The last integral in the above equation was considered to be the total energy associated with the wave by Ko and Kuehl (1979) and by Miles (1978). It is easy to prove that this integral is an integral invariant of the cylindrical KdV equation, but its physical interpretation is not clear.

By the same argument as was used in the previous sections, the energy conservation principle becomes

\[
\frac{\partial}{\partial \tau} \int_0^\infty r \eta^2 \, dr \equiv 0. \tag{3-16}
\]

Multiplying the cylindrical KdV equation (2-30) by \( 2r \eta \) and integrating over the interval \( 0 < r < \infty \), we obtain
\[ \frac{3}{\partial t} \int_0^\infty r^2 \, dr + \int_0^\infty n^2 \, dr + \int_0^\infty r \frac{\partial n^2}{\partial r} \, dr \]

\[ + \varepsilon \int_0^\infty r \frac{\partial n^3}{\partial r} \, dr + \frac{\mu^2}{3} \int_0^\infty r \frac{\partial^3 n}{\partial r^3} \, dr = 0(\varepsilon^2). \quad (3-17) \]

Assuming that \( n, n_r, \) and \( n_{rr} \) vanish at infinity and using the boundary condition at \( r = 0 \) \( \left( \frac{\partial n(0,t)}{\partial r} = 0 \right) \), we have

\[ \int_0^\infty r \frac{\partial n^2}{\partial r} \, dr = - \int_0^\infty n^2 \, dr, \]

\[ \int_0^\infty r \frac{\partial n^3}{\partial r} \, dr = - \int_0^\infty n^3 \, dr, \]

\[ \int_0^\infty r n \frac{\partial^3 n}{\partial r^3} \, dr = \frac{3}{2} \int_0^\infty \left( \frac{\partial n}{\partial r} \right)^2 \, dr. \]

Substituting the above integrals into equation (3-17), we have

\[ \frac{3}{\partial t} \int_0^\infty r \, n^2 \, dr = \varepsilon \left( \int_0^\infty n^3 \, dr - \frac{\mu^2}{2\varepsilon} \int_0^\infty \left( \frac{\partial n}{\partial r} \right)^2 \, dr \right) + 0(\varepsilon^2). \quad (3-18) \]

Therefore, if the terms of the order of \( \varepsilon \) are neglected, the total energy is conserved.

From equation (3-18) we note that if only \( \varepsilon^2 \) terms are neglected, the energy-conservation principle requires that

\[ \int_0^\infty n^3 \, dr = \frac{\mu^2}{2\varepsilon} \int_0^\infty \left( \frac{\partial n}{\partial r} \right)^2 \, dr. \quad (3-19) \]
Thus, we can conclude that a single negative-hump wave or a train of only negative waves is not a possible solution of the cylindrical KdV equation, because this type of solution does not satisfy equation (3-19).
CHAPTER IV

APPROXIMATE SOLUTION OF THE CYLINDRICAL KORTEWEG-de VRIES EQUATION (OUTER SOLUTION)

The KdV equation has been widely used by many investigators as an accurate model for the propagation (in one direction) of two-dimensional long waves in water of constant depth. Great interest in this equation has been generated by the exact solutions of the KdV equation found by Miura et al. (1968). As a natural extension, a new interest in the three-dimensional long waves has been revived in recent years.

In this chapter, the behavior of the cylindrical solitary wave for incoming and outgoing propagation in water of constant depth is investigated.

4.1 Outgoing wave

A cylindrical outgoing solitary wave propagating in water of constant depth has been shown to be governed by the following equation:

\[
\frac{\partial n}{\partial t} + \frac{1}{2} \frac{n}{r} + \left(1 + \frac{3}{2} \epsilon n\right) \frac{\partial n}{\partial r} + \frac{\mu^2}{6} \frac{\partial^3 n}{\partial r^3} = 0(\mu^4). \tag{4-1}
\]

In this section we shall find an approximate solution of the above equation when the initial wave profile corresponds to a planar solitary wave located sufficiently far from the origin, that is
\[ \eta = \text{sech}^2 \left( \frac{\sqrt{5} \varepsilon}{2 \mu} [r - r_0 - (1 + \frac{\varepsilon}{2})(t - t_0)] \right) \]  

(4-2)

where the initial time is taken to be \( t_0 \) instead of zero and the initial wave crest is located at \( r_0 \) \((r_0 \gg 1)\).

The leading term of equation (4-1) is

\[ \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial r} + \frac{1}{2} \frac{\eta}{r} = 0, \]  

(4-3)

which admits a solution of the form

\[ \eta = \frac{f(r-t)}{r^{1/2}} \]  

(4-4)

The above solution has a maximum wave amplitude which decreases like \( r^{-1/2} \) as \( r \) increases and the wave shape is given by \( f(r-t) \). This solution has to satisfy the condition (3-6) implied by the mass conservation principle discussed in Chapter III. The function \( f(r-t) \) corresponding to a solitary wave produced by a point acoustic source was found by Lamb (1902).

The cylindrical KdV equation in terms of the variables \( \sigma \) and \( \tau \) is given by equation (2-29). Introducing the following change of variables (see Cumberbatch, 1978)

\[ \nu = \eta \left( \frac{r}{r_0} \right)^{1/2} \]  

(4-5)

and

\[ \nu = r_0^{1/2} \left( r^{1/2} - r_0^{1/2} \right) \]  

(4-6)
into equation (2-29), we obtain

\[
\frac{\partial v}{\partial \nu} + 3v \frac{\partial v}{\partial \sigma} + \frac{\mu}{3\epsilon} \left( \frac{\tau}{\tau_0} \right)^{1/2} \frac{\partial^3 v}{\partial \sigma^3} = 0(\mu^2). \tag{4-7}
\]

The above equation belongs to a family of KdV equations with variable coefficients. For this kind of equations an exact analytical solution is found only when \( \tau \) is not too different from \( \tau_0 \) by a perturbation expansion around \( \tau_0 \) as was done by Johnson (1973) and by Ko and Kuehl (1978). Some approximate solutions for the main part of the wave were given by Ogino and Takeda (1978), Cumberbatch (1978) and Miles (1978, 1979), but all of these have some discrepancies from the known numerical solutions.

We shall assume a solution of the form

\[
v = G\left[ \frac{(\sigma - \sigma_0) - \nu}{w(\nu)} \right], \tag{4-8}
\]

where \( w \) is an arbitrary function of \( \nu \) yet to be determined. Hence, the wave profile is

\[
\eta = \left( \frac{\tau_0}{\tau} \right)^{1/2} G\left[ \frac{(\sigma - \sigma_0) - \nu}{w(\nu)} \right], \tag{4-9}
\]

which indicates that the amplitude and width change during the development of the wave. The wave amplitude function is

\[
a(\tau) = \left( \frac{\tau_0}{\tau} \right)^{1/2}, \tag{4-10}
\]
which is in accordance with the leading order solution (4-4), and agrees with Chwang and Wu's (1976) numerical solution as is shown later in this section.

Substituting (4-8) into equation (4-7) and assuming \( \left| \frac{d w}{d \nu} \right| \) to be of the order of \( O(\varepsilon) \), we obtain

\[
\frac{\mu^2}{3\varepsilon} \left( \frac{I}{t_0} \right)^{1/2} G''' + w^2(\nu) G'(3G-1) = O(\mu^2),
\]

(4-11)

where

\[
G' = \frac{dG}{dn} \quad \text{and} \quad \Omega = \frac{(\sigma - \sigma_0) - \nu}{w(\nu)}.
\]

(4-12)

Equation (4-11) allows the following separation of variables:

\[
w^2(\nu) = \left( \frac{I}{t_0} \right)^{1/2},
\]

(4-13)

\[
\frac{\mu^2}{3\varepsilon} G''' + G'(3G-1) = O(\mu^2).
\]

(4-14)

Equation (4-13) gives the width function for the wave profile \( n \). Equations (4-10) and (4-13) satisfy the relation between the wave amplitude and width found experimentally by Hershkowitz and Romesser (1974) and numerically by Maxon and Viecelli (1974),

\[
w a^{1/2} = 1.
\]

(4-15)
Integrating equation (4-14) and neglecting terms of $O(\mu^2)$, we have

$$\frac{\mu^2}{3\epsilon} G'' + \frac{3}{2} G^2 - G + B = 0. \quad (4-16)$$

Multiplying (4-16) by $G'$ and integrating once more with respect to $\Omega$, we have

$$\frac{\mu^2}{3\epsilon} \frac{(G')^2}{G^3} + G^2 + 2BG + D = 0, \quad (4-17)$$

where $B$ and $D$ are arbitrary constants of integration.

If $G$, $G'$ and $G''$ diminish to zero as $| (\sigma - \sigma_o) - \nu |$ goes to infinity, the constants $B$ and $D$ will be zero. Therefore equation (4-17) becomes

$$(G')^2 = 3 \frac{\epsilon}{\mu^2} [G^2(1 - G)]. \quad (4-18)$$

The solution of equation (4-18) is

$$G = \text{sech}^2 \left[ \frac{\sqrt{3\epsilon}}{2\mu} \frac{(\sigma - \sigma_o) - \nu}{w(\nu)} \right]. \quad (4-19)$$

By equations (2-21), (4-9), (4-13) and (4-19), the wave profile $\eta$ can be written as

$$\eta = \left( \frac{t_o}{t} \right)^{1/2} \text{sech}^2 \left[ \frac{\sqrt{3\epsilon}}{2\mu} \left( \frac{t_o}{t} \right)^{1/4} \frac{(r - r_o)}{(t - t_o) - (t - t_o)} \right] - \epsilon \left( \frac{t_o}{t} \right)^{1/2} \left( t - t_o \right)^{1/2} \left( t - t_o \right)^{1/2} \left[ (r - r_o) - (t - t_o) \right]. \quad (4-20)$$
The position for the maximum \( \eta \) at a given time \( t_{\text{max}} \) is found at

\[
 r_{\text{max}} = r_o + (t_{\text{max}} - t_o) + \varepsilon t_o^{1/2} (t_{\text{max}}^{1/2} - t_o^{1/2}), \tag{4-21}
\]

and the phase velocity of the wave is

\[
 C = \frac{dr}{dt} \bigg|_{r=r_{\text{max}}} = 1 + \frac{\varepsilon}{2} \left( \frac{t_o}{t_{\text{max}}} \right)^{1/2}. \tag{4-22}
\]

For simplification we let \( r_o = t_o \). Therefore, the maximum value of the wave amplitude becomes

\[
 \eta_{\text{max}} = \left( \frac{t_o}{r_{\text{max}}} \right)^{1/2} + O(\varepsilon), \tag{4-23}
\]

as is expected from the leading order solution. The phase velocity can be written as

\[
 C = 1 + \frac{\varepsilon}{2} \left( \frac{t_o}{r_{\text{max}}} \right)^{1/2} + O(\varepsilon^2). \tag{4-24}
\]

The behavior of the wave predicted by equations (4-23) and (4-24) is in agreement with the numerical solution of Chwang and Wu (1976) in the outer region.

In deriving equation (4-11) we made the assumption that

\[
 \frac{dw}{dv} \ll 1.
\]
Substituting equation (4-13) and (2-21) into the above inequality and noting that \( t_0 = r_0 \), we obtain

\[
t \gg \frac{1}{\frac{4}{2} \epsilon \frac{4}{3} r_0^3}. \tag{4-25}
\]

Hence, our approximate solution holds for almost all the time if the initial wave is located very far away from the origin for fixed values of \( \epsilon \).

4.2 Incoming wave

Axisymmetric long waves propagating in water of constant depth have been shown to be governed by the cylindrical Boussinesq equations (2-17) and (2-18). For permanent incoming waves we define a new pair of variables \( \zeta \) and \( \tau \) by

\[
\zeta = r + t, \quad \tau = \epsilon t. \tag{4-26}
\]

In terms of these new variables, equations (2-17) and (2-18) become

\[
\epsilon \eta_\tau + \eta_\zeta + \frac{\epsilon}{\epsilon_\zeta - \epsilon_\tau} [u + \left( \frac{\epsilon \zeta - \epsilon_\tau}{\epsilon} \right) u_\zeta] +
\]

\[
\epsilon \eta u + (\epsilon \zeta - \epsilon_\tau)(\eta u)_\zeta = 0(\mu^4), \tag{4-27}
\]
\[
\varepsilon u_t + u_\zeta + \varepsilon u u_\zeta + \eta_\zeta - \frac{\mu^2}{3} \left\{ \frac{\varepsilon}{\varepsilon \zeta - \tau} \left[ \frac{\varepsilon}{\varepsilon \zeta - \tau} u \right] \zeta_\zeta \right\} - \frac{\varepsilon u}{3} = 0(\mu^4). \tag{4-28}
\]

Subtracting the last two equations and neglecting terms of the order of \(0(\mu^4)\), we get

\[
\varepsilon \left[ \frac{\partial}{\partial \tau} (\eta - u) - u \frac{\partial u}{\partial \zeta} + \frac{u}{\varepsilon \zeta - \tau} + \frac{\eta}{\partial \zeta} \right]
\]

\[
(\eta u) + \frac{\mu^2}{3} \eta \frac{\partial^3 u}{\partial \zeta^3} = 0(\mu^4). \tag{4-29}
\]

To the lowest order, neglecting terms of the order of \(0(\varepsilon)\) and \(0(\mu^2)\), we have from equations (4-27) and (4-28)

\[
\eta = - u + 0(\varepsilon). \tag{4-30}
\]

Substituting equations (4-30) into equation (4-29) and neglecting terms of the order of \(0(\mu^4)\), we obtain

\[
\frac{\partial \eta}{\partial \tau} - \frac{3}{2} \eta \frac{\partial^2 \eta}{\partial \zeta^2} + \frac{1}{2} \eta \frac{\partial^3 \eta}{\partial \zeta^3} = 0. \tag{4-31}
\]

With \(\zeta = -s\), equation (4-31) becomes

\[
\frac{\partial \eta}{\partial \tau} + \frac{3}{2} \eta \frac{\partial \eta}{\partial s} + \frac{1}{2} \eta \frac{\partial^2 \eta}{\partial s^2} + \frac{\mu^2}{6 \varepsilon} \frac{\partial^3 \eta}{\partial s^3} = 0, \tag{4-32}
\]
which is the same as the cylindrical KdV equation for outgoing waves (2-29) with \( \sigma \) replaced by \( s \). Therefore the solution for an incoming solitary wave may be obtained from (4-20) by replacing \( \sigma \) by \( s \), i.e., replacing \( r \) by \(-r\). Hence, by means of the identity \( \text{sech}^2(-x) = \text{sech}^2(x) \), the solution becomes

\[
\eta = \frac{t_0^{1/2}}{t^{1/2}} \sech^2 \left( \frac{\sqrt{3} \epsilon}{2 \mu} \frac{t_0}{t} \right) \left[ (t-r_0) + (t-t_0) + \epsilon t_0^{1/2} \right] (t^{1/2} - t_0^{1/2}) \].
\]  

(4-33)

As in the previous section, we let \( t_0 = -r_0 \) for simplicity. Thus, the position for the maximum value of \( \eta \) is found at

\[
r_{\text{max}} = -t_{\text{max}} - \epsilon [(t_0 t_{\text{max}})^{1/2} - t_0].
\]  

(4-34)

The maximum wave amplitude is

\[
\eta_{\text{max}} = \left( \frac{r_0}{r_{\text{max}}} \right)^{1/2} + O(\epsilon),
\]  

(4-35)

and the phase velocity is

\[
C = - \left[ 1 + \frac{\epsilon}{2} \left( \frac{r_0}{r_{\text{max}}} \right) \right] + O(\epsilon^2).
\]  

(4-36)

Equation (4-33) holds when

\[
|t| >> \frac{1}{2^4 \frac{4}{\epsilon} \frac{3}{r_0}}.
\]  

(4-37)
As $t$ goes to zero, the wave approaches the origin and the cylindrical KdV equation is no longer valid due to the reflection and focusing of the wave. Near the origin the cylindrical Boussinesq equations are the correct governing equations. An analytical solution of this focusing process is presented in Chapter V, section 5-2, based on an inner-outer expansion technique.

4.3 Family of solutions

In the past, almost all the analytical solutions for cylindrical solitary waves contain a sech$^2$ function, the main difference between them is how the wave amplitude and wave width change during propagation.

A cylindrical outgoing solitary wave in water of constant depth is governed by equation (2-29). A general solution of (2-29) has the form

$$
\eta = a(\tau) \ G[\frac{(\sigma - \sigma_0) - g(\tau)}{w(\tau)}], \quad (4-38)
$$

where $a(\tau)$, $g(\tau)$ and $w(\tau)$ are undetermined functions of $\tau$.

Substituting equation (4-38) into equation (2-29), yields

$$
\frac{\mu}{3\varepsilon} G^{''''} + aw^2 G'(3G - \frac{2}{a} \frac{dg}{d\tau}) - 2G' = 0.
$$

Substituting

$$
\frac{(\sigma - \sigma_0) - g(\tau)}{w(\tau)}(w^2 \frac{dw}{d\tau} + \frac{w^3}{a} - G(2 \frac{da}{d\tau} + \frac{a}{\tau})) = 0(\varepsilon). \quad (4-39)
$$

The shape of the main part of the wave is, in general, given by
\[ G = \text{sech}^2 \left( \frac{\sqrt{5\varepsilon}}{2\mu} \left[ \frac{(\sigma - \sigma_0) - g(\tau)}{w(\tau)} \right] \right), \]  

(4-40)

The above equation is an exact solution of the differential equation

\[ \frac{2}{3\varepsilon} G''' + G'(3G-1) = 0 \]  

(4-41)

subject to the boundary conditions

\[ G(\pm \infty) = G'(\pm \infty) = G''(\pm \infty) \equiv 0. \]  

(4-42)

Now, equation (4-39) would be identical to equation (4-41) if and only if

\[ \frac{2}{a} \frac{d\sigma(\tau)}{d\tau} \equiv 1, \]  

(4-43)

\[ a \frac{w^2}{w} \equiv 1, \]  

(4-44)

\[ w^2 \frac{dw(\tau)}{d\tau} = 0(\varepsilon), \]  

(4-45)

and

\[ \frac{w^3}{a} \left( 2 \frac{da(\tau)}{d\tau} + \frac{a}{\tau} \right) = 0(\varepsilon). \]  

(4-46)

Equation (4-44) is in accordance with the relation (4-15).

Assuming an amplitude function
\[ a(t) = \left( \frac{\tau_0}{t} \right)^n, \quad (4-47) \]

We find from equation (4-44) that

\[ w(\tau) = \left( \frac{\tau}{\tau_0} \right)^{n/2}. \quad (4-48) \]

By equations (4-47) and (4-48), conditions (4-45) and (4-46) reduce to

\[ w^2 \frac{dw}{d\tau} = \frac{n}{2} \frac{\tau}{(\frac{\tau}{\tau_0})^{(3/2)n}} = O(\varepsilon), \]

\[ \frac{3}{a} \left( 2 \frac{da}{d\tau} + \frac{a}{\tau} \right) = (1-2n) \frac{\tau}{(\frac{\tau}{\tau_0})^{(3/2)n}} = O(\varepsilon). \]

This required condition from the last two equations, \( \left( \frac{\tau}{\tau_0} \right)^{(3/2)n} \frac{1}{\tau} = O(\varepsilon) \), is satisfied for all values of \( n \) if \( \tau \) is a very large number (\( \tau = O(1/\varepsilon) \)). Therefore, based on the above assumption we obtain a family of solutions for the main part of the wave

\[ n = \left( \frac{\tau_0}{\tau} \right)^n \text{sech}^2 \left\{ \frac{\sqrt{3\varepsilon}}{2\mu} \left( \frac{\tau_0}{\tau} \right)^{n/2} [\sigma - \sigma_0 - g(\tau)] \right\}. \quad (4-49) \]

To find \( g(\tau) \) we integrate equation (4-43),

\[ g(\tau) - g(\tau_0) = \int_{\tau_0}^{\tau} \frac{1}{2} a(\tau') d\tau'. \quad (4-50) \]

Substituting equation (4-47) into equation (4-50), we have
g(τ) = \frac{1}{2(1-n)} τ^n_o (τ(1-n) - τ_o (1-n)) + g(τ_o).

(4-51)

Therefore, from equations (4-49) and (4-51) we note that with n = 1/2 the result is our solution, equation (4-20), and with n = 2/3 the solution is Cumberbatch's (1978) solution.

As in previous sections we shall let t_o = r_o for simplicity, then the position of the maximum value of \eta is found at

\[ r_{\text{max}} = t_{\text{max}} + \frac{ε \cdot t^n_o}{2(1-n)} (t(1-n) - t_o (1-n)), \]

since \( g(τ_o) \) is equal to zero from the initial condition (4-2).

Therefore, the maximum value of \( \eta \) is

\[ \eta_{\text{max}} = \left( \frac{r_o}{r_{\text{max}}} \right)^n + O(ε), \]

and the phase velocity is

\[ C = 1 + \frac{ε}{2} \left( \frac{r_o}{r_{\text{max}}} \right)^n + O(ε^2). \]

Of the above family of solutions we expect that only one has a physical meaning. If we let an outgoing wave propagate towards infinity the wave amplitude decreases and the wave width increases. Eventually, it will reach a region in which the propagation is governed by a linear non-dispersive equation. The only solution of the above family which matches the linear solution (4-4) at infinity is our approximate solution (4-20).
CHAPTER V
INNER SOLUTION OF THE BOUSSINESQ EQUATIONS

In the previous chapter we found an approximate solution of the cylindrical KdV equation. When the wave approaches the origin, this approximate solution is no longer valid, because of the wave reflection during the focusing process, during which the entire Boussinesq equations are the correct governing equations.

In this chapter an analytical solution of the cylindrical Boussinesq equations will be presented based on the inner-outer expansion technique. We first solve the problem of the reflection of a planar solitary wave at a vertical wall in section 5.1 in order to develop the necessary mathematical technique for analyzing the cylindrical solitary wave.

5.1 Reflection of a planar solitary wave

Two-dimensional long waves propagating in water of constant depth are governed by the Boussinesq equations,

\[ \eta_t + [(1 + \epsilon \eta)u]_x = 0(\epsilon \mu^2, \mu^4), \quad (5-1) \]

and

\[ u_t + \epsilon uu_x + \eta_x - \frac{\mu^2}{3} u_{xxt} = 0(\epsilon \mu^2, \mu^4). \quad (5-2) \]
For waves moving in the positive x direction only, equations (5-1) and (5-2) may be reduced to the KdV equation,

\[ \eta_t + \eta_x + \frac{3}{2} \varepsilon \eta \eta_x + \frac{\mu^2}{6} \eta_{xxx} = 0, \]  

(5-3)

where the terms of the order of \( O(\varepsilon \mu^2, \mu^4) \) are neglected. This equation has an exact analytical solution in the form of a wave of permanent shape, namely a solitary wave solution,

\[ \eta = \text{sech}^2 \left( \frac{\sqrt{3\varepsilon}}{2\mu} \left[ (x - x_0) - (1 + \frac{\varepsilon}{2})t \right] \right). \]  

(5-4)

At the time \( t = 0 \), the maximum amplitude of the solitary wave occurs at \( x = x_0 \). By equations (5-1) and (5-2), \( u \) is related to \( \eta \) as

\[ u = \eta + O(\varepsilon). \]  

(5-5)

Based on the Boussinesq equations, the second approximation to the solitary wave solution is given by Laitone (1960)(see appendix A) as

\[ \eta = (1 - \frac{3}{4} \varepsilon \tanh^2 X) \text{sech}^2 X + O(\varepsilon^2), \]  

(5-6a)

where

\[ X = \frac{\sqrt{3\varepsilon}}{2\mu} \left[ (x - x_0) - (1 + \frac{\varepsilon}{2})t \right]. \]  

(5-6b)

For waves propagating in the negative x direction, the KdV equation becomes
\[-\eta_t + \eta_x + \frac{3}{2} \epsilon \eta_x + \frac{\mu^2}{6} \eta_{xxx} = 0.\]

(5-7)

The corresponding solitary wave solution is

\[\eta = \text{sech}^2 \left( \frac{\sqrt{3\epsilon}}{2\mu} \right) \left[ (x - x_o) + (1 + \frac{\epsilon}{2})t \right].\]

(5-8)

and \(u\) is related to \(\eta\) as

\[u = -\eta + O(\epsilon)\].

(5-9)

In this section an analytical solution of the Boussinesq equations will be presented for the reflection of a solitary wave at a vertical wall, based on the inner-outer expansion technique. The incident solitary wave is assumed to propagate in the negative \(x\) direction,

\[\eta_i = \text{sech}^2 \left\{ \frac{\sqrt{3\epsilon}}{2\mu} \left[ (x - x_o) + (1 + \frac{\epsilon}{2})t \right] \right\}\]

(5-10)

with \(x_o >> 1\) and the vertical wall is located at \(x = 0\).

The Boussinesq equations (5-1) and (5-2) can be combined to give

\[\eta_{tt} - \eta_{xx} + \frac{\mu^2}{3} \eta_{xxxx} + \epsilon (\eta u)_x = 0(\epsilon \mu^2, \mu^4)\].

(5-11)

If we introduce the following variables (outer variables)
\[ \alpha = (x - x_0) + (1 + \varepsilon/2)t, \quad (5.12a) \]

and

\[ \beta = (x - x_1) - (1 + \varepsilon/2)t, \quad (5.12b) \]

where \(x_1\) is an unknown constant, the derivatives \(\frac{\partial}{\partial t}, \frac{\partial^2}{\partial t^2}, \frac{\partial}{\partial x}\) and \(\frac{\partial^2}{\partial x^2}\) become

\[
\frac{\partial}{\partial t} = (1 + \varepsilon/2) \left( \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta} \right),
\]

\[
\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} - 2 \frac{\partial^2}{\partial \alpha \partial \beta} + \varepsilon \left( \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right) - 2 \frac{\partial^2}{\partial \alpha \partial \beta} + 0(\varepsilon^2),
\]

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta},
\]

and

\[
\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + 2 \frac{\partial^2}{\partial \alpha \partial \beta}.
\]

Assuming a perturbation solution of equation (5.11) as

\[ n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots, \quad (5.13) \]
and

\[ u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ---, \]  

(5-14)

we find that the zeroth-order equation is

\[ -4 \frac{\partial^2 \eta_0}{\partial \alpha \partial \beta} = 0. \]  

(5-15)

Integrating the above equation, we obtain

\[ \eta_0 = \eta_{0_1}(\alpha) + \eta_{0_r}(\beta), \]  

(5-16)

and

\[ u_0 = u_{0_1}(\alpha) + u_{0_r}(\beta), \]  

(5-17)

where the subscripts i and r denote the incident and reflected waves respectively. From the initial condition (5-10), we have

\[ \eta_{0_1}(\alpha) = \text{sech}^2 \left\{ \frac{\sqrt{3} \varepsilon}{2\mu} \alpha \right\}, \]  

(5-18)

where \( \eta_{0_1} \) is an exact solution of the differential equation (5-7).

Also \( u_{0_1} \) is related to \( \eta_{0_1} \) by

\[ u_{0_1} = -\eta_{0_1}. \]  

(5-19)
To find the inner expansion of the Boussinesq equations, we first expand the region near \( x = 0 \) by

\[
x = \varepsilon \xi. \tag{5-20}
\]

With this change of variable the Boussinesq equations become

\[
\varepsilon [N_t + \langle UN \rangle_\xi] + U_\xi = 0 (\varepsilon^2 \mu^2, \varepsilon \mu^4), \tag{5-21}
\]

and

\[
\varepsilon [U_t + UU_\xi] + N_\xi - \frac{1}{3} \frac{\mu^2}{\varepsilon} U_\xi U_{\xi t} = 0 (\varepsilon^2 \mu^2, \varepsilon \mu^4), \tag{5-22}
\]

where \( N \) and \( U \) denote \( \eta \) and \( u \) respectively in the inner region.

We now assume a perturbation solution of the type

\[
N = N_0 + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots, \tag{5-23}
\]

and

\[
U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 \ldots. \tag{5-24}
\]

Substituting equations (5-23) and (5-24) into equations (5-21) and (5-22), we find the zeroth-order problem to be

\[
N_0 = 0, \tag{5-25}
\]

and
\[ U_0 \xi = 0. \quad (5-26) \]

Integrating these two equations, we obtain

\[ N_0 = f(t), \quad (5-27) \]

and

\[ U_0 = g(t), \quad (5-28) \]

where \( f(t) \) and \( g(t) \) are arbitrary functions of \( t \) only. From the boundary condition at the wall that the velocity at \( x = 0 \) is zero, we require \( g(t) \) to be zero for all values of \( t \). Therefore

\[ U_0 \equiv 0. \quad (5-29) \]

Corresponding to a zeroth-order solution in the inner region of velocity zero \((U_0 \equiv 0)\) we must have a zeroth-order solution in the outer region with a velocity which goes to zero as \( x \) goes to zero. Thus, by equations (5-16) to (5-18), we determine \( \eta_{or}(\beta) \) in the outer region as

\[ \eta_{or}(\beta) = \text{sech}^2 \left( \frac{\sqrt{3\varepsilon}}{2\mu} \beta \right), \quad (5-30) \]

where \( \beta \) is given by (5-12b) with \( x_1 = -x_0 \), and \( U_{or} \) is related to \( \eta_{or} \) by

\[ u_{or} = \eta_{or}. \quad (5-31) \]
Therefore the zeroth-order outer solution is a linear combination of an incident wave whose initial position at \( t = 0 \) is at \( x = x_o \) and a reflected wave whose initial position is at an imaginary point located at \( -x_o \). Thus,

\[
\eta_o = \text{sech}^2 \left( \frac{\sqrt{3\varepsilon}}{2\mu} \alpha \right) + \text{sech}^2 \left( \frac{\sqrt{3\varepsilon}}{2\mu} \beta \right),
\]

and

\[
u_o = \text{sech}^2 \left( \frac{\sqrt{3\varepsilon}}{2\mu} \beta \right) - \text{sech}^2 \left( \frac{\sqrt{3\varepsilon}}{2\mu} \alpha \right).
\]

The behavior of these solutions when \( x \) goes to zero is

\[
\lim_{x \to 0} u_o = \lim_{\xi \to \infty} u_o = 0, \quad \lim_{x \to 0} \eta_o = 2 \text{sech}^2 \theta,
\]

where

\[
\theta = \frac{\sqrt{3\varepsilon}}{2\mu} [(1 + \varepsilon/2)t - x_o].
\]

From the matching condition that

\[
\lim_{\xi \to \infty} \eta_o = \lim_{x \to 0} \eta_o,
\]

we obtain \( N_o \) as

\[
N_o = f(t) = 2 \text{sech}^2 \theta.
\]

From equations (5-21) to (5-24) the first order inner problem becomes

\[
N_{o\xi} + (U N_o)_{\xi} + U_{1\xi} = 0,
\]
\[ U_{0t} + U_{0ot} + N_{1\xi} - \frac{1}{3} \frac{\mu}{\varepsilon} U_{1\xi \xi t} = 0. \]  

(5-37)

Substituting equation (5-29) into equations (5-36) and (5-37), we have

\[ N_{0t} + U_{1\xi} = 0, \]  

(5-38)

\[ N_{1\xi} - \frac{1}{3} \frac{\mu}{\varepsilon} U_{1\xi \xi t} = 0. \]  

(5-39)

Integrating these two equations, we obtain

\[ U_1 = - \xi N_{0t}, \]  

(5-40)

\[ N_1 = \frac{\mu}{3\varepsilon} U_{1\xi t} + H(t), \]  

(5-41)

where we used the boundary condition at \( x = 0 \) and \( H(t) \) is an arbitrary function of time. The derivative of equation (5-38) with respect to \( t \) gives

\[ U_{1\xi t} = - N_{0tt}. \]  

(5-42)

Substituting this equation into equation (5-41), we obtain

\[ N_1 = H(t) - \frac{\mu}{3\varepsilon} N_{0tt}. \]  

(5-43)

The second-order inner problem is obtained by substituting equations (5-23) and (5-24) into equations (5-21) and (5-22). Hence
\[ N_1 + (U_0' - N_1') \xi + (U_1 N_0) \xi + U_2 \xi = 0, \tag{5-44} \]

\[ U_1 + U_0' U_1 \xi + U_1 U_0 \xi + N_2 \xi - \frac{1}{3} \frac{\mu^2}{\varepsilon} U_2 \xi t = 0. \tag{5-45} \]

Using the zeroth-order solution (5-29) and (5-35), we can reduce (5-39) and (5-40) to

\[ N_1 + N_0 U_1 \xi + U_2 \xi = 0, \tag{5-46} \]

and

\[ U_1 + N_2 \xi - \frac{1}{3} \frac{\mu^2}{\varepsilon} U_2 \xi t = 0. \tag{5-47} \]

Substituting equation (5-38) into equation (5-46), we have

\[ U_2 \xi = N_0 N_1 - N_1 t. \tag{5-48} \]

Therefore \( U_2 \xi \) is a function of time only by means of (5-35) and (5-43). Hence equations (5-40) and (5-47) yield

\[ N_2 \xi = \xi N_{ott}. \tag{5-49} \]

The mathematical condition for \( N \) to be a maximum at \( x = 0 \) is

\[ N_\xi = 0 \text{ and } N_{\xi \xi} < 0 \text{ at } x = 0. \]
In the present case, we have

\[ N_\xi = N_{0\xi} + e N_{1\xi} + e^2 N_{2\xi} = e^2 N_{0tt}, \]  

(5-50)

and

\[ N_{\xi\xi} = e^2 N_{0tt}. \]  

(5-51)

From equation (5-50) we find that up to the order of \( O(e^2) \), \( N_\xi \) is always zero at the origin \( x = 0 \). Referring to Figure 3, we note that the peak of the wave reaches the origin when \( N_{0tt} \) changes from positive to negative at \( t = t_1 \) and it leaves the origin when \( N_{0tt} \) changes from negative to positive at \( t = t_2 \). Between \( t_1 \) and \( t_2 \) the crest of the wave remains at the origin. From the above argument, we have

\[ N_{0tt} = 0, \text{ at } t = t_1 \text{ and } t = t_2. \]  

(5-52)

By equation (5-11) to (5-14) and the zeroth-order outer solution (5-32) and (5-33), the first-order outer problem becomes

\[ 4 \frac{\partial^2 \eta_1}{\partial \alpha \partial \beta} = 2 \frac{\partial \eta_{0i}}{\partial \alpha} \frac{\partial \eta_{0r}}{\partial \beta} + \eta_{0i} \frac{\partial^2 \eta_{0r}}{\partial \beta^2} + \eta_{0r} \frac{\partial^2 \eta_{0i}}{\partial \alpha^2}. \]  

(5-53)

Equation (5-33) may be written as

\[ \frac{\partial^2 \eta_1}{\partial \alpha \partial \beta} = \frac{\partial^2}{\partial \alpha \partial \beta} \left\{ \frac{1}{2} \eta_{bi} \frac{\partial^2 \eta_{0r}}{\partial \beta^2} + \frac{1}{4} \int_{-\infty}^{\alpha} \eta_{0r} (\xi) d\xi + \int_{-\infty}^{\alpha} \eta_{bi} (\xi) d\xi \right\}, \]  

(5-54)
where the choice of the lower limits of integration is based on the given initial condition (5-10).

Integrating equation (5-54) twice leads to

\[
\eta_1 = \frac{1}{2} \eta_i \eta_o + \frac{1}{4} \left[ \frac{\partial \eta}{\partial \beta} \right]_{\infty}^\alpha \eta_o (\xi) \, d\xi
\]

\[
+ \frac{\partial \eta}{\partial \alpha} \left[ \eta_o (\xi) \, d\xi \right]_{\infty}^\beta + h_1(\alpha) + h_2(\beta),
\]

(5-55)

where \( h_1(\alpha) \) and \( h_2(\beta) \) are arbitrary functions of \( \alpha \) and \( \beta \) respectively.

Substituting equations (5-18) and (5-30) into equation (5-55), we have

\[
\eta_1 = \frac{1}{2} \operatorname{sech}^2 \left( \frac{\sqrt{3\epsilon}}{2\mu} \alpha \right) \operatorname{sech}^2 \left( \frac{\sqrt{3\epsilon}}{2\mu} \beta \right) - \frac{1}{2}
\]

\[
\operatorname{sech}^2 \left( \frac{\sqrt{3\epsilon}}{2\mu} \beta \right) \tanh \left( \frac{\sqrt{3\epsilon}}{2\mu} \beta \right) \left[ \tanh \left( \frac{\sqrt{3\epsilon}}{2\mu} \alpha \right) + 1 \right]
\]

\[- \frac{1}{2} \operatorname{sech}^2 \left( \frac{\sqrt{3\epsilon}}{2\mu} \alpha \right) \tanh \left( \frac{\sqrt{3\epsilon}}{2\mu} \alpha \right) \left[ \tanh \left( \frac{\sqrt{3\epsilon}}{2\mu} \beta \right) - 1 \right]
\]

\[
+ h_1(\alpha) + h_2(\beta).
\]

(5-56)

Now, \( h_1(\alpha) \) and \( h_2(\beta) \) are determined by the second approximation of a solitary wave given by Laitone (1960) (5-6a) such that the initial solitary wave is given by equation (5-10). Therefore,
\[ \eta = \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \left[ 1 - \frac{3}{4} \epsilon \tanh^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \right] + \]

\[ \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) \left[ 1 - \frac{3}{4} \epsilon \tanh^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) \right] + \frac{\epsilon}{2} \left( \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \right. \]

\[ \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) - \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \tanh \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \left[ \tanh \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) - 1 \right] \]

\[ - \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) \tanh \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) \left[ \tanh \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \right) \]

\[ + 1 \right) \right] + O(\epsilon^2). \] (5-57)

By means of Taylor's series about \( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \) and \( \frac{\sqrt{3} \epsilon}{2\mu} \beta \), equation (5-57) can be written as

\[ \eta = \text{sech}^2 \left[ \frac{\sqrt{3} \epsilon}{2\mu} \left( \alpha + \delta_1 \right) \right] + \text{sech}^2 \left[ \frac{\sqrt{3} \epsilon}{2\mu} \left( \beta + \delta_2 \right) \right] - \frac{3}{4} \epsilon \tanh^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) - \frac{3}{4} \epsilon \tanh^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) \]

\[ \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) + \frac{\epsilon}{2} \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \alpha \right) \text{sech}^2 \left( \frac{\sqrt{3} \epsilon}{2\mu} \beta \right) + O(\epsilon^2), \] (5-58)

where

\[ \frac{\delta_1}{\mu} = \frac{1}{2} \sqrt{\epsilon} \left[ \tanh(\frac{\sqrt{3} \epsilon}{2\mu} \beta) - 1 \right], \] (5-59)

and

\[ \frac{\delta_2}{\mu} = \frac{1}{2} \sqrt{\epsilon} \left[ \tanh(\frac{\sqrt{3} \epsilon}{2\mu} \alpha) + 1 \right]. \] (5-60)
Since the initial wave (5-10) is located far away from the origin \(x_o \gg 1\), \(\beta_{t=0} = (x + x_o)\) is always a positive large number. Thus, we obtain, from equations (5-58) to (5-60),

\[
\lim_{t \to 0} \eta = \text{sech}^2\left(\frac{\sqrt{3\epsilon}}{2\mu} \alpha\right) \left[1 - \frac{3}{4} \epsilon \tanh^2\left(\frac{\sqrt{3\epsilon}}{2\mu} \alpha\right)\right] \tag{5-61}
\]

for \(x \geq 0\).

At \(t = \frac{2 x_o}{(1+\epsilon/2)} \gg 1\), \(\alpha = (x + x_o)\) is always a positive large number. Hence, we find from equations (5-58) to (5-60) that for \(x \geq 0\),

\[
\lim_{t=2x_o/(1+\epsilon/2) \to \infty} \eta = \text{sech}^2\left[\frac{\sqrt{3\epsilon}}{2\mu} \left(\beta + \delta_2\right)\right] - \frac{3}{4} \epsilon \tanh^2\left(\frac{\sqrt{3\epsilon}}{2\mu} \beta\right)
\]

\[
\text{sech}^2\left(\frac{\sqrt{3\epsilon}}{2\mu} \beta\right), \tag{5-62}
\]

\[
\delta_2 = k \frac{\epsilon}{\sqrt{3}}. \tag{5-63}
\]

Therefore, at \(t = \frac{2 x_o}{(1+\epsilon/2)}\), the maximum amplitude of the reflected wave occurs at \(x_m\), where

\[
x_m = x_o - k h \frac{\epsilon}{\sqrt{3}} + O(\epsilon^2), \tag{5-64}
\]

and the phase shift \(\Delta x\) is

\[
\Delta x = x_o - x_m = k h \frac{\epsilon}{\sqrt{3}} + O(\epsilon^2). \tag{5-65}
\]
Equation (5-65) is in agreement with the result given by Byatt-Smith (1971), Oikawa and Yajima (1973) and Su and Mirie (1980). In figure 4 we plot the location of the wave crest versus time. This figure shows a definition sketch of the phase shift $\Delta x$.

By equation (5-62) we can conclude that after the collision the shape of the wave changes and the maximum amplitude is less than the initial one (see figure 5).

The behavior of the first-order outer solution when $x$ goes to zero is

$$\lim_{x \to 0} \eta_1 = \frac{1}{2} \text{sech}^2 \theta \left( \text{sech}^2 \theta - \tanh^2 \theta + 2 \tanh \theta \right), \quad (5-66)$$

where $\theta$ is defined by (5-34).

Substituting equation (5-35) into equation (5-43), we have

$$N_1 = H(t) + \text{sech}^4 \theta - 2 \text{sech}^2 \theta \tanh \theta. \quad (5-67)$$

Now, the matching condition requires

$$\lim_{\xi \to \infty, \epsilon \to 0} \frac{N - N_0}{\epsilon} = \lim_{x \to 0, \epsilon \to 0} \frac{\eta - \eta_0}{\epsilon}. \quad (5-68)$$

By equations (5-66), (5-67) and (5-68), we conclude that

$$H(t) = \frac{\text{sech}^2 \theta}{2} \left( 3 \tanh^2 \theta - \text{sech}^2 \theta + 2 \tanh \theta \right).$$
Therefore,

\[ N = 2 \sech^2 \theta + \frac{\varepsilon}{2} \sech^2 \theta (1 + 2 \tanh \theta - 2 \tanh^2 \theta) + O(\varepsilon^2). \quad (5-69) \]

The maximum value of \( N \) given by equation (5-69) is

\[ N_{\text{max}} = 2 + \frac{\varepsilon}{2} + O(\varepsilon^2), \quad (5-70) \]

which occurs at \( \theta = \varepsilon/4 \) or

\[ t = (x_0 + \frac{\mu}{2} \sqrt{\frac{\varepsilon}{3}})/(1+\varepsilon^2). \quad (5-71) \]

The value of \( N_{\text{max}} \) given by equation (5-70) is also in agreement with the known result.

Equation (5-35) and (5-52) lead to

\[ \sech^2 \theta \tanh^2 \theta - \frac{1}{2} \sech^4 \theta = 0, \quad (5-72) \]

at \( t = t_1 \) and \( t = t_2 \). Therefore

\[ \theta = \mp 0.658 \text{ at } t = t_1 \text{ and } t = t_2. \quad (5-73) \]

By equation (5-34), the duration that the crest of the wave remains at the origin \( x = 0 \) is

\[ \Delta t = t_2 - t_1 = \frac{2.63 \mu}{\sqrt{3} \varepsilon (1+\varepsilon^2/2)}. \quad (5-74) \]
In terms of the original physical variables, equation (5-74) can be written as

\[
\Delta t^* \left( \frac{g^*}{h^*} \right)^{1/2} = \frac{1.52}{\sqrt{\varepsilon(1 + \varepsilon/2)}}.
\]  

(5-75)

From the above equation we deduce that \( \Delta t^* \) goes to infinity as \( a_o \) goes to zero. A simple reasoning to support the present conclusion may be derived from the kinematic free-surface condition at \( x = 0 \).

The kinematic free-surface condition is

\[
\frac{\partial N^*}{\partial t^*} + \frac{\partial \phi^*}{\partial x^*} \frac{\partial N^*}{\partial x^*} = W^* \text{ at } z^* = N^*.
\]

At the origin \( x = 0 \),

\[
\frac{\partial \phi^*}{\partial x^*} \equiv 0.
\]

Therefore at the origin we have

\[
W^* = \frac{\partial N^*}{\partial t^*} \text{ at } z^* = N^*.
\]

Now, from our solution we know that \( N^* \) is proportional to \( a_o \). Therefore at the origin \( W^* \) is proportional to \( a_o^{3/2} \). Then

\[
\Delta t^* \sim N^*^{-1/2} \quad W^* \sim a_o^{3/2}.
\]
which is in agreement with the behavior predicted by equation (5-75). In figure 6 we plot $\Delta t^* \sqrt{g/h}$ versus $\epsilon$ predicted by equation (5-75) as well as the only existing experimental value found in the literature (Maxworthy, 1976).

5.2 Focusing and reflection of a cylindrical solitary wave

Cylindrical long waves propagating in water of constant depth are governed by the cylindrical Boussinesq equations (2-17) and (2-18). To find the inner expansion of this system of equations we introduce an inner variable $\xi$ by

$$r = \epsilon \xi.$$  \hspace{1cm} (5-76)

Substituting (5-76) into (2-17) and (2-18), we obtain

$$\epsilon [N_t + \frac{1}{\epsilon} (\xi U N)_\xi] + \frac{1}{\epsilon} (\xi U)_\xi = 0(\epsilon^2 u^2, \epsilon^4 u^4),$$ \hspace{1cm} (5-77)

and

$$\epsilon [U_t + U U_x] + N_x - \frac{1}{3} \frac{u^2}{\epsilon} \left[ \frac{1}{\epsilon} (\xi U)_\xi \right]_{\xi t} = 0(\epsilon^2 u^2, \epsilon^4 u^4),$$ \hspace{1cm} (5-78)

where $N$ and $U$ denote $\eta$ and $u$ respectively in the inner region.

We now assume a perturbation solution of the type

$$N = N_0 + \epsilon N_1 + \epsilon^2 N_2 + \cdots,$$ \hspace{1cm} (5-79)
and

\[ U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots. \]  \hfill (5-80)

Substituting equations (5-79) and (5-80) into equations (5-77) and (5-78), we have the zeroth-order problem as

\[ (\xi U'_0)_{\xi} = 0, \]  \hfill (5-81)

and

\[ N_0_{\xi} = 0. \]  \hfill (5-82)

Integrating equations (5-81) and (5-82), we have

\[ N_0 = f(t) \quad \text{and} \quad U_0 = g(t)/\xi, \]  \hfill (5-83)

where \( f(t) \) and \( g(t) \) are arbitrary functions of \( t \) only. From the boundary condition at the origin that the velocity must be finite, we require \( g(t) \) to be zero. Therefore,

\[ U_0 = 0. \]  \hfill (5-84)

Corresponding to a zeroth-order solution in the inner region of velocity zero we must have a zeroth-order solution in the outer region with a velocity which goes to zero as \( r \) goes to zero. Hence, the zeroth-order
outer solution is

\[ \eta_o = \eta_{o_i} + \eta_{o_r}, \]  

(5-85)

and

\[ u_o = u_{o_i} + u_{o_r}, \]  

(5-86)

where the incident component \( \eta_{o_i} \) is given by (4-33) with \( u_{o_i} = - \eta_{o_i} \) and the outgoing component \( \eta_{o_r} \) is given by

\[
\eta_{o_r} = \left( \frac{t_0}{t} \right)^{1/2} \operatorname{sech} \left\{ \frac{\sqrt{3\varepsilon}}{2\mu} \left( \frac{t_0}{t} \right)^{1/4} \left[ (r + r_o) - (t - t_o) \right]^{1/2} \right\}^{1/2} - \varepsilon \frac{t_o}{t} \left( t - t_o \right). \]

with \( u_{o_r} = \eta_{o_r} \).

Consequently, the zeroth-order outer solution is a linear combination of an incident wave whose initial position at \( t = t_o \) is at \( r = r_o \) and a reflected wave whose initial position at \( t = t_o \) is at an imaginary point located at \(-r_o\). The behavior of this solution when \( r \) goes to zero is

\[ \lim_{r \to 0} u_o = 0, \]

(5-87)

and
\[ \lim_{r \to r_0} \eta_0 = 2(t_0^0)^{1/2} \text{sech}^2\left\{\frac{\sqrt{3} \varepsilon}{\mu} \left(1 + \frac{t_0^0}{t}\right)^{1/4}\right\} [t + \varepsilon t_0^{1/2}] \], \quad (5-88) \]

where we have assumed \( t_0 = -r_0 \) without loss of generality. From the matching condition

\[ \lim_{\xi \to \infty} N_0 = \lim_{r \to r_0} \eta_0, \]

we obtain \( N_0 \) as

\[ N_0 = f(t) = 2(t_0^0)^{1/2} \text{sech}^2\left\{\frac{\sqrt{3} \varepsilon}{2\mu} \left(\frac{t_0^0}{t}\right)^{1/4}\right\} [t + \varepsilon t_0^{1/2}] \]

\[ (t - t_0^{1/2}) \}. \quad (5-89) \]

From equations (5-77) to (5-80) and (5-84), the first-order inner problem becomes

\[ N_0 t + \frac{1}{\xi} [\xi U_1]_{\xi} = 0, \quad (5-90) \]

\[ N_1 \xi - \frac{1}{3} \frac{\mu^2}{\varepsilon} \left[\frac{1}{\xi} (\xi U_1)_{\xi}\right]_{\xi} t = 0. \quad (5-91) \]

Integrating equation (5-91) and using equation (5-90), we have

\[ N_1 = H(t) - \frac{1}{3} \frac{\mu^2}{\varepsilon} N_0_{tt}, \quad (5-92) \]
where \( H(t) \) is an arbitrary function of \( t \). Integrating equation (5-90) and using the boundary condition at \( r = 0 \), we find

\[
U_1 = - N_0 \xi \frac{\xi}{2}.
\]  
(5-93)

The second-order inner problem is obtained by substituting equations (5-79) and (5-80) into (5-77) and (5-78) and by using equations (5-84) and (5-89). Hence

\[
N_{1t} + N_0 \frac{1}{\xi} (\xi U_1) + \frac{1}{\xi} (\xi U_2) = 0,
\]  
(5-94)

\[
U_{1t} + N_2 \frac{2}{\xi} - \frac{1}{3} \frac{\mu}{\varepsilon} \left[ \frac{1}{\xi} (\xi U_2) \right] \xi t = 0.
\]  
(5-95)

By equations (5-90) and (5-94), we have

\[
\frac{1}{\xi} (\xi U_2) = N_0 N_0 \xi t - N_{1t}.
\]

The right-hand side of the above equation is a function of time only. Therefore,

\[
\left[ \frac{1}{\xi} (\xi U_2) \right] \xi \equiv 0,
\]

and equations (5-93) and (5-95) yield

\[
N_2 \xi = N_0 \xi t \frac{\xi}{2}.
\]  
(5-96)
By the same argument as was used in section 5.1, the peak of the wave reaches the origin at \( t = t_1 \) and leaves the origin at \( t = t_2 \), and

\[
N_{o_{tt}} = 0 \text{ at } t = t_1 \text{ and } t = t_2. \tag{5-97}
\]

In order to compare the present result with Chwang and Wu's (1976) numerical solution, we shall define the dimensionless variables \( \tilde{N} \) and \( \tilde{t} \) by

\[
\tilde{N} = N*/h \text{ and } \tilde{t} = t* (g/h)^{1/2}. \tag{5-98}
\]

In terms of these dimensionless variables equation (5-89) becomes

\[
\begin{align*}
\tilde{N}_0 &= 2\varepsilon \left( \frac{t_0}{t} \right)^{1/2} \operatorname{sech}^2 \left( \frac{\sqrt{3\varepsilon}}{2} \left( \frac{t_0}{t} \right) \right)^{1/4} [t + \varepsilon t_0]^{1/2} \\
&\quad \left( \frac{\tilde{t} - \tilde{t}_0}{\tilde{t}} \right)]^2.
\end{align*} \tag{5-99}
\]

For values of \( \varepsilon = 0.1 \) and \( \tilde{t}_0 = -30 \) (where \( \tilde{t}_0 = - \tilde{r}_0 \)), equation (5-99) gives a maximum value of \( \tilde{N}_{o_{\max}} \) = 0.55 at \( \tilde{t} = -3.7 \). Substituting (5-99) into (5-97), we find that, for \( \varepsilon = 0.1 \) and \( \tilde{t}_0 = -30 \), the peak of the wave reaches the origin at \( \tilde{t}_1 = -5.4 \) with a wave amplitude of \( \tilde{N}_o = 0.40 \) and it leaves the origin at \( \tilde{t}_2 = -1.9 \). Thus, the wave crest remains at the origin for 3.5 units of time.

Transforming Chwang and Wu's (1976) numerical solution into our frame of reference, we find that their initial condition corresponds to an incoming solitary wave with \( \varepsilon = 0.1 \) and \( \tilde{r}_0 = 30 \). Their numerical
results show that the maximum wave amplitude grows initially like $r^{-1/2}$ as $r$ decreases. At $\tilde{t} = -5$, the crest of the wave arrives at the center $r = 0$, reaching an amplitude of 0.42. For the next three units of time, the crest of the wave remains at $r = 0$, while the amplitude keeps increasing until it reaches a maximum value of 0.59 at $\tilde{t} = -3$; then it starts to decrease. At $\tilde{t} = -2$, the crest begins to move away from the center and the wave changes its direction. Therefore, by comparing the present analytical result with Chwang and Wu's (1976) solution we can conclude that our perturbation solution is in good agreement with the numerical result.
CHAPTER VI

CONCLUSIONS

The major objective of this study is to investigate analytically the propagation and focusing of a cylindrical solitary wave in water of constant depth. To achieve this goal, we applied an inner-outer expansion technique to the cylindrical Boussinesq equations.

The most important conclusions drawn from this study are the following:

1. A single positive wave or a train of solely positive waves is not a possible solution of the cylindrical KdV equation since these solutions do not satisfy the principle of mass conservation. This conclusion is due to the three-dimensionality and not to the non-linearity of the problem.

2. The total energy associated with the presence of the wave (the total energy of the system minus the potential energy of the fluid at rest) is

\[ E^* = \frac{2\pi \rho gh^2}{k_0^2} \left\{ \int_{-\infty}^{\infty} \tau \eta^2 \, d\tau + \varepsilon \int_{-\infty}^{\infty} \sigma \eta^2 \, d\tau \right\} + \tau \int_{-\infty}^{\infty} \eta^2 \, d\sigma \} \]

(6-1)
Based on the principle of energy conservation, a single negative-hump wave or a train of solely negative waves is not a possible solution of the cylindrical KdV equation.

3. The maximum wave amplitude of an outgoing cylindrical solitary wave decreases like $r^{-1/2}$ as $r$ increases, this feature is in accordance with the leading-order linear solution.

4. The maximum wave amplitude of an incoming cylindrical solitary wave initially grows like $r^{-1/2}$ as $r$ decreases. As the wave approaches the origin the maximum wave amplitude grows somewhat faster than $r^{-1/2}$ due to the focusing process.

5. The phase velocity for an outgoing cylindrical solitary wave is

$$C = 1 + \frac{\varepsilon}{2} \left( \frac{r}{r_{\text{max}}} \right)^{1/2}. \quad (6-2)$$

This phase velocity approaches the phase velocity of the linear solution as $r$ goes to infinity, as is expected from the matching condition at infinity.

6. The phase velocity for an incoming cylindrical solitary wave is initially given by the negative of equation (6-2). When the wave is near the origin, its phase velocity increases quite rapidly. This remarkable feature is attributed to the rapid increase in the wave amplitude near $r = 0$ due to focusing.

7. The inner-outer expansion method used to study the reflection of a planar solitary wave due to a vertical wall leads to the following two conclusions:
(a) From the outer solution we note that the reflected wave corresponding to an initial incoming wave (5-10) is given by

\[ \eta = \text{sech}^2 \left[ \frac{\sqrt{3} \varepsilon}{2 \mu} \ (\beta + \delta_2) \right] - \frac{3}{4} \varepsilon \ \tanh^2 \left( \frac{\sqrt{3} \varepsilon}{2 \mu} \ \beta \right) \]

\[ \text{sech}^2 \left( \frac{\sqrt{3} \varepsilon}{2 \mu} \ \beta \right), \quad (6-3) \]

where \( \beta = (x + x_0) - (1 + \varepsilon/2)t \), and \( \delta_2 \) is a phase shift due to the interaction process, given by

\[ \delta_2 = \mu \sqrt[3]{\varepsilon} \cdot \quad (6-4) \]

From equations (6-3) and (6-4), we deduce that after the collision the shape of the wave changes and the maximum wave amplitude is less than the initial one (see figure 5).

(b) From the inner solution we conclude that the maximum amplitude at the wall is greater than twice the initial wave amplitude. This maximum amplitude occurs at \( \theta = \varepsilon/4 \) or

\[ t = (x_0 + \frac{\mu}{2} \sqrt[3]{\varepsilon})/(1 + \varepsilon/2). \quad (6-5) \]

The wave crest remains at the wall for an interval of \( \Delta t \) known as the phase lag which is inversely proportional to the initial amplitude (see figure 6),

\[ \Delta t = \frac{2.63 \mu}{\sqrt{3} \varepsilon \ (1 + \varepsilon/2)} \cdot \quad (6-6) \]
8. The phase lag for a cylindrical solitary wave, $\Delta t = t_2 - t_1$, is determined by equations (5-97) and (5-99). For given values of $\epsilon = 0.1$ and $\tilde{t}_0 = -30$, we find that the crest of the wave remains at the origin $r = 0$ for 3.5 units of time, which is in good agreement with the numerical solution of Chwang and Wu (1976).
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APPENDIX A

THE SECOND APPROXIMATION OF A SOLITARY WAVE

The expansion method introduced by Friedrichs (1948) for shallow-water waves of large wavelength was used by Keller (1948) to obtain the first approximation of a finite-amplitude solitary wave of Boussinesq (1872) and Rayleigh (1876), as well as periodic waves of permanent type, corresponding to cnoidal waves of Korteweg and de Vries (1895).

Laitone (1960) extended Friedrichs' method to obtain the complete second approximation for cnoidal waves, the solitary wave is the infinite-period limit of the cnoidal waves. Unlike the first approximation, the vertical motions cannot be neglected and the pressure variation is no longer hydrostatic. Grinshaw (1970) obtained a solitary-wave solution up to the order of $O(\varepsilon^3)$ using the same shallow-water expansion method.

In this appendix we shall derive the second approximation of a solitary wave. The flow is assumed to be irrotational and the fluid is inviscid and incompressible. The vertical z axis has its origin at the still water level, the displacement of the free surface from the still water level is $\eta^*(x^*, t^*)$, and the solid bottom is $z^* = -h$ (constant). The fluid is supposed to be unbounded in the x direction.

We introduce the dimensionless variables by
\[ x^* = h x, \quad z^* = h z, \quad t^* = t \sqrt{\frac{h}{g}}, \]

\[ \eta^* = h \eta \text{ and } \phi^* = \phi h \sqrt{gh}. \]

The equations of motion for the velocity potential \( \phi(x,z,t) \) are

\[ \phi_{xx} + \phi_{zz} = 0 \text{ in } -1 < z < \eta, \quad (A-1) \]
\[ \phi_z = 0 \text{ at } z = -1, \quad (A-2) \]
\[ \eta_t + \phi_x \eta_x - \phi_z = 0 \text{ at } z = \eta, \quad (A-3) \]
\[ \phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + \eta = 0 \text{ at } z = \eta. \quad (A-4) \]

We shall seek a solution of equations (A-1) through (A-4) for which \( \eta \) and \( \phi \) are functions of \( z \) only, and the phase function is

\[ \sigma = k(x^* - c^* t^*) = \mu(x-ct), \quad (A-5) \]

where \( k \) (wave-number) and \( c^* \) (wave speed) are constants. Thus we seek a solution of the form

\[ \eta = \eta(\sigma) \text{ and } \phi = \phi(\sigma,z). \quad (A-6) \]

Substitution of equation (A-5) into the equations of motion gives

\[ \mu^2 \phi_{\sigma\sigma} + \phi_{zz} = 0 \text{ in } -1 < z < \eta, \quad (A-7) \]
\[ \phi_z = 0 \text{ at } z = -1, \quad (A-8) \]
\[-\mu c \eta \sigma + \mu^2 \frac{\partial^2 \eta}{\partial \sigma^2} + \frac{\partial \phi}{\partial z} = 0 \text{ at } z = \eta, \quad (A-9)\]
\[-\mu c \phi \sigma + \frac{1}{2} (\mu^2 \phi^2_{\sigma} + \phi^2_z) + \eta = 0 \text{ at } z = \eta. \quad (A-10)\]

Without loss of generality, we may select the origin of \(\sigma\) at the crest of the wave so that

\[\eta(0) = \epsilon \text{ and } \eta_0(0) = 0. \quad (A-11)\]

Now, following Friedrichs (1948), we assume that \(\eta, \phi, c\) and \(\mu\) each has a power series expansion in terms of the small parameter \(\epsilon = a/h\) in the form of

\[\eta(\sigma) = \epsilon \{\eta_0(\psi) + \epsilon \eta_1(\psi) + \cdots\}; \quad (A-12)\]
\[\phi(\sigma, z) = \epsilon^{1/2} \{\phi_0(\psi, z) + \epsilon \phi_1(\psi, z) + \cdots\}; \quad (A-13)\]
\[c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \cdots, \quad (A-14)\]
\[\mu = \mu_0 \{1 + \epsilon \mu_1 + \cdots\}, \quad (A-15)\]

and

\[\psi = \frac{\epsilon^{1/2}}{\mu_0^{1/2}} \sigma, \quad (A-16)\]

where \(\eta_i's, \phi_i's\) and their derivatives vanish as \(|\sigma|\) goes to infinity.
The function $\psi$ is suggested by the Boussinesq's solitary wave profile.

Substitution of equations (A-12), (A-13) and (A-15) into equations (A-7) and (A-8) gives

$$\phi_0(\psi, z) = F_0(\psi),$$

$$\phi_1(\psi, z) = F_1(\psi) - \frac{1}{2} (z + 1)^2 F_0''(\psi),$$

$$\phi_2(\psi, z) = F_2(\psi) - \frac{1}{2} (z+1)^2 F_1''(\psi) +$$

$$\frac{1}{4!} (z+1)^4 F_0''''(\psi) - \nu_1 (z+1)^2 F_0'', \text{ etc.}$$

The $\epsilon$-terms of the free surface boundary conditions (A-9) and (A-10) become,

$$- c_0 \eta_0' + F_0'' = 0, \quad \eta_0 - c_0 F_0' = 0. \quad (A-17)$$

Therefore

$$c_0 = \pm 1, \quad \eta_0 = \pm F_0'. \quad (A-18)$$

For waves propagating in the positive x direction, positive signs in (A-18) should be chosen. The $\epsilon^2$-terms of (A-9) and (A-10) are

$$- \eta_1' + F_1'' = (c_1 + \mu_1) \eta_0' - F_0' \eta_0' - 2\mu_1 F_0'' - \eta_0 F_0'' + \frac{1}{6} F_0''', \text{ etc.}$$
\[ \eta_1 - F'_{1} = -\frac{1}{2} (F'_o)^2 + (c_1 + \mu_1)F'_o - \frac{1}{2} F''_o. \]

Since these two equations must be compatible, we have

\[ \frac{1}{3} \eta_0''' + \frac{3}{2} (\eta_0^2)' - 2c_1 \eta_0' = 0. \quad (A-19) \]

The solution of this equation which satisfies equation (A-11) and vanishes as \(|\sigma|\) goes to infinity is

\[ \eta_0 = \text{sech}^2 \left( \frac{3}{4} \right)^{1/2} \psi \text{ and } c_1 = 1/2. \quad (A-20) \]

This is the Boussinesq solitary wave. Application of the free surface boundary conditions yields, at each stage, a pair of equations. The compatibility, at each stage, takes the form of

\[ \frac{1}{3} \eta_i'' + 3\eta_0 \eta_i - \eta_i = H_i \quad (i = 1, 2, \ldots), \quad (A-21) \]

where \(H_i\) is known in terms of \(\eta_0\), \(c_{i+1}\) and \(\mu_i\). The homogeneous part of equation (A-21) has \(\eta_0\) as a solution. Hence (A-21) may be integrated once to give

\[ \frac{1}{3} \left( \eta_i' \eta_0' - \eta_i \eta_0'' \right) = \int_{\psi}^{\infty} \eta_0' H_i \, d\xi. \quad (A-22) \]

The application of equation (A-11) implies that
\[ \int_{\eta_0}^{\infty} H_1 \, d\xi = 0 \quad (i = 1, 2, \ldots), \] (A-23)

and this determines \( c_{i+1} \). To keep the expansion well ordered, we now impose the condition that \( \eta_i \) should vanish as \( |\sigma| \) goes to infinity, this determines \( \mu_1 \). Hence, equation (A-22) may then be integrated to give \( \eta_1 \).

The compatibility condition for \( \eta_2 \) and \( F_2 \) is

\[ \frac{1}{3} \eta_1^{\prime\prime} + 3 \eta_0 \eta_1 - \eta_1 = H_1, \] (A-24)

where

\[ H_1 = \eta_0 \left( 2c_2 - \frac{19}{20} - 2\mu_1 \right) + \frac{\eta_0^2}{2} (3 + 3\mu_1) - \frac{3}{2} \eta_0^3. \] (A-25)

Substitution of (A-25) into (A-23) gives \( c_2 = -3/20 \). Integrating equation (A-22) for \( i = 1 \) with \( H_1 \) given by (A-25), we obtain that

\[ \mu_1 = -\frac{5}{8}, \] (A-26)

\[ \eta_1 = -\frac{3}{4} \text{sech}^2 \left[ \left( \frac{3}{4} \right)^{1/2} \psi \right] \tanh^2 \left[ \left( \frac{3}{4} \right)^{1/2} \psi \right], \] (A-27)

where

\[ \left( \frac{3}{4} \right)^{1/2} \psi = \sqrt{\frac{3\varepsilon}{2\mu_0}} \mu(x - ct), \] (A-28)

\[ \mu = \mu_0 \left( 1 - \frac{5}{8} \varepsilon + \cdots \right), \] (A-29)

\[ c = \left( 1 + \frac{1}{2\varepsilon} - \frac{3}{20} \varepsilon^2 + \cdots \right). \] (A-30)
Figure 2. Wave elevation $\eta$ versus $t$ given by the linear nondispersive theory.
Figure 3. Detail of the wave profile in the inner region during the reflection of a planar solitary wave due to a vertical wall.
Figure 4. Location of the wave crest versus time for a planar solitary wave.
Figure 5. Detail of the outer solution for the reflection of a planar solitary wave.
Figure 6. Magnitude of $\Delta t^*(g/h)^{1/2}$ versus $\varepsilon$ for the reflection of a planar solitary wave.