NALED ICE GROWTH

by

Gerald A. Schohl and Robert Ettema

Sponsored by

National Science Foundation
Grant CEE-81-09252

IIHR Report No. 297

Iowa Institute of Hydraulic Research
The University of Iowa
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ABSTRACT

Based on theoretical formulation and dimensional analysis, supported by the results of laboratory experiments, a theory and a detailed description of naled ice growth are presented. The theory, concepts, and data should be of interest to engineers concerned with the effects of naleds (also referred to as aufeis or icings) on engineering works.

The growth of a two-dimensional, or laterally confined (flume), naled is shown to depend primarily on seven, independent, dimensionless parameters. The early, two-dimensional, phase of naled ice growth depends on only four of the seven parameters. During this phase of growth, a naled consists of a mixture of ice and water, or ice-water slush, forming on a frigid base. The influence of two of the three remaining parameters is not felt until after a transition time has passed. This transition time apparently coincides with the beginning of the processes by which the ice-water slush on a naled's surface freezes solid. After a slush layer on a naled begins to freeze solid, a new slush layer forms over its frozen surface. The continuing, cyclic process by which slush layers form and eventually freeze results in the ice laminations that are a feature of naled ice. The influence of the seventh governing parameter, a Reynolds number, cannot be discerned in the laboratory data.
ACKNOWLEDGEMENTS

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The authors wish to thank the sources of the various photographs of naleds in nature that are included in Chapters II and III. Kevin Carey and the U.S. Army Cold Regions Research and Engineering Laboratory (CRREL) kindly permitted the reproduction of Figures 2, 8, and 13 from "Icings Developed from Surface Water and Ground Water" (Carey 1973) for use as Figures 2.1, 3.7, and 2.2. Also, Figures 3.1, 3.4, and 3.6 are adaptations of Figures 15 and 16 from Carey. The U.S. Geological Survey, through Chet Zenone, gave permission for the reproduction of Figures 8, 10 and 26 from "Icings along the Trans-Alaska Pipeline Route" (Sloan, Zenone, and Mayo 1976), which, herein, are Figures 3.3, 3.4, and 3.5, respectively. Zenone was even helpful enough to provide a print from the original negative of the photograph in Figure 3.8. The photograph in Figure 2.3a was reproduced from Permafrost in Canada: Its Influence on Northern Development (Brown 1970) with the permission of the University of Toronto Press and, in a letter from G.H. Johnston, of the National Research Council of Canada. The photograph in Figure 2.3b, from Geotechnical Engineering for Cold Regions (Andersland and Anderson 1978), is included with the permission of McGraw-Hill, and with the personal permission of O.B. Andersland.
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LIST OF SYMBOLS

\( a \) cross-sectional area of unfrozen valley alluvium
\( A \) cross-sectional area of stream channel
\( A_b \) outside surface area of plexiglas box (see Appendix B)
\( A_{ws} \) area of water surface in plexiglas box (see Appendix B)
\( b \) thickness of snow cover
\( B \) area of river drainage basin
\( c \) heat conductivity of valley alluvium under a naled
\( C \) coefficient in (2.3)
\( C_1 \) coefficient in (8.8)
\( C_2 \) coefficient in (8.9)
\( C_w \) specific heat of water
\( d \) depth of water or ice-water slush (see Figure 4.1)
\( d^* \) \( d \) normalized by \( L \) (see (4.35))
\( d_{co} \) critical depth in wide channel with discharge \( q_o \) per unit width
\( f \) designator for "function of"
\( F \) temperature difference from the freezing temperature of water
\( F_x \) longitudinal dispersion coefficient (see Appendix A)
\( g \) acceleration of gravity
\( h \) elevation of water or ice-water slush surface (see Figure 4.5)
\( h^* \) \( h \) normalized by \( L \)
\( h_b \) coefficient of heat transfer from naled into flume floor
\( h_{ba} \) coefficient of heat transfer for determining \( \phi_{ba} \) (see Appendix B)
h<sub>wa</sub> coefficient of heat transfer from wet naled surface into the air
h<sub>wi</sub> coefficient of heat transfer from water flow into underlying ice
I "intensity" of naled development (see (2.1) and (2.2))
k coefficient in (2.3)
k<sub>i</sub> thermal conductivity of ice
K<sub>1</sub> coefficient in (B.4)
K<sub>2</sub> coefficient in (B.5)
L streamwise spread length of a naled (see Figure 1.1)
L<sub>e</sub> equilibrium length of a two-dimensional naled (see (4.31))
L<sub>eo</sub> equilibrium length using $\phi_{10}$ rather than $\phi_1$
L<sub>1</sub> equilibrium length if $\phi_{wa}$ is neglected (see (4.45))
L<sub>s</sub> equilibrium length if $\phi_i$ is neglected (see (4.32))
L latent heat of fusion for water
m porosity of porous ice
m<sub>x</sub> areal concentration of pores in x-direction
m<sub>y</sub> areal concentration of pores in y-direction
M* normalized total mass per unit width of naled (see (7.3))
M<sub>i</sub> total ice mass per unit width of naled
M<sub>i</sub>* normalized $M_i$ (see (7.5))
M<sub>wa</sub> water mass per unit width of naled or in plexiglas box (in (B.6))
M<sub>i</sub> normalized unfrozen water mass per unit width of naled
p pressure
q water discharge per unit width (see (4.16))
q* q normalized by $q_o$ (see (4.35))
q<sub>0</sub> q at x = 0
Q water discharge
\( Q_o \) Q at \( x = 0 \) (see Figure 1.1)

\( R_e \) Reynolds number (see (4.54))

\( R_x \) x-directed force on a unit weight of water due to porous ice

\( \overline{R_x} \) depth-averaged value of \( R_x \) (see (4.24))

\( R_y \) y-directed force on a unit weight of water due to porous ice

\( s \) total thickness of a naled (see Figure 4.6)

\( S \) naled surface area

\( S_o \) slope of frigid base under naled

\( S_s \) naled surface area per unit source water discharge (see (2.4))

\( t \) time

\( t^* \) \( t \) normalized by \( t_e \) (see (4.37))

\( t_o \) time that corresponds to \( m = 1 \) in (4.30)

\( t_1 \) time at beginning of integration interval in (B.3)

\( t_2 \) time at end of integration interval in (B.3)

\( t_e \) time scale based on \( q_o \) and \( l_e \) (see (4.37))

\( t_{eo} \) time scale based on \( q_o \) and \( l_{eo} \)

\( t_{g(x)} \) the time when a naled's length of spread reaches position \( x \)

\( t_s \) time scaled based on \( q_o \) and \( l_s \) (see (6.1))

\( T \) local water temperature

\( \overline{T} \) depth-averaged water temperature (see (A.9))

\( T_{1/3} \) water temperature \( d/3 \) below water surface (see Appendix B)

\( T_{2/3} \) water temperature \( 2d/3 \) below water surface (see Appendix B)

\( T_a \) air temperature (see Figure 4.1)

\( T_b \) flume coolant temperature (see Figure 1.1)

\( T_f \) freezing temperature of water

\( T_s \) water temperature at water surface (see Figure 4.1)
$T_w$  bulk water temperature (see (A.10))

$T_{w1}$  bulk water temperature at time $= t_1$ (see Appendix B)

$T_{w2}$  bulk water temperature at time $= t_2$ (see Appendix B)

$u$  local water velocity in $x$-direction (see Figure 4.5)

$\bar{u}$  depth-averaged value of $u$ (see (4.15))

$v$  local water velocity in $y$-direction (see Figure 4.5)

$V$  volume of a naled

$V_s$  naled volume per unit source water discharge (see (2.5))

$V_{si}$  volume of ice platelets per unit surface area of slush (see (4.6))

$w$  width of naled, stream, or flume

$W$  width of river valley

$x$  horizontal (longitudinal) coordinate

$x^*$  $x$ normalized by $l_e$ (see (4.35))

$X$  depth of frozen soil on the banks of a stream

$y$  vertical coordinate

$y_b$  elevation of water-ice interface (see Figure 4.5)

$y_g$  elevation of ground surface under naled (see Figure 4.5)

$\alpha$  coefficient in (B.8) and (B.9)

$\alpha_i$  thermal diffusivity of ice

$\alpha_w$  thermal diffusivity of water

$\alpha_{tx}$  thermal turbulent diffusivity in $x$-direction

$\alpha_{ty}$  thermal turbulent diffusivity in $y$-direction

$\beta$  momentum correction factor

$\Delta$  small increment of distance; e.g., $\Delta x$

$\varepsilon_x$  eddy viscosity coefficient in $x$-direction
\( \varepsilon_y \)  eddy viscosity coefficient in y-direction

\( \zeta \)  thickness of a lamination in naled ice (see Figure 4.6)

\( \eta \)  thickness of naled under the surface slush (see Figure 4.1)

\( \eta^* \)  \( \eta \) normalized by \( \ell_e \)

\( \nu \)  kinematic viscosity of water

\( \xi \)  mean temperature correction factor

\( \rho_i \)  mass density of ice

\( \rho_w \)  mass density of water

\( \tau_o \)  bed shear stress

\( \phi_i \)  heat flux from water-ice interface into ground (see Figure 4.1)

\( \phi_{ba} \)  heat flux from outer surface of plexiglas box (see Appendix B)

\( \phi_{wa} \)  heat flux from naled surface to air (see Figure 4.1)

\( \phi_{wa}^* \)  normalized \( \phi_{wa} \) (see (4.53))

\( \phi_{wi} \)  heat flux from water into ice over which it flows (see Figure 4.1)

\( \phi_x \)  x-directed heat flux from porous ice boundary (see Figure 4.4)

\( \phi_y \)  y-directed heat flux from porous ice boundary (see Figure 4.4)

\( \dot{\psi} \)  local viscous dissipation function

\( \ddot{\psi} \)  depth-averaged viscous dissipation function (see (A.11))

\( \dot{\psi}_r \)  ratio of \( \phi_{wa} \) to \( (\phi_{wa} + \phi_{i}) \) (see (4.52))

\( \psi \)  general dependent variable (see (4.49))

\( \psi^* \)  normalized general dependent variable (see (4.51))
CHAPTER I

INTRODUCTION

The major contributions of this study are the developments of a theory and a detailed description of naled ice growth. These are based on theoretical formulation and dimensional analysis supported by the results of extensive laboratory experiments.

An important part of this study is a review of the notably scant, and dispersed, literature on naled ice growth. The review served to guide both the design of the laboratory experiments and the theoretical formulation.

Naleds (also referred to as aufeis or icings) are spreading and thickening ice accretions that grow in cold, winter air when a shallow flow of water streams over a river ice cover, or over frozen ground, and freezes progressively to it. A naled can form initially on any surface, but its subsequent growth always proceeds as a progressive accretion of ice.

For over a century, naleds have been of interest and concern to scientists and engineers living in cold regions. Naleds may cause a variety of engineering problems. For example, they may block drainage facilities, and thereby cause subsequent spring flooding and washouts of embankments; they may inundate roads, railroads, and airfields causing them to become unusable; they may engulf bridges spanning streams; they
may threaten floodplain communities and individual homes; and they may disrupt the operation of tunnels and mines. Naleds were a major concern in the selection of the route for the trans-Alaska pipeline.

During the present study, small-scale naleds were observed and monitored as they grew under conditions of steady discharge and heat flux rates in a refrigerated flume located in the Low Temperature Flow Facility at the Iowa Institute of Hydraulic Research. These flume naleds are herein termed "laterally confined" naleds because they were confined by the side-walls of the flume and an upstream gate so that they could spread only along the downstream direction.

A simplified, or idealized, laterally confined naled is illustrated schematically in Figure 1.1, in which a naled is depicted as a formation of ice covered by a shallow flow of water. Because its longitudinal cross section is invariant across the width of the flume, and its shape is defined by a typical longitudinal, two-dimensional cross section, the naled shown in Figure 1.1 may be considered to be two-dimensional. The following terms can be used to describe the geometry of a two-dimensional naled: naled length, \( \ell(t) \); thickness of naled that underlies the surface flow, \( \eta(x,t) \); and the depth of water flow, \( d(x,t) \). The definition of a two-dimensional naled is useful for describing the energy budget for naled ice growth, and for deriving conservation equations for naled ice growth. The flume naleds were two-dimensional during their early growth, but their later growth was influenced by three-dimensional, or boundary, effects. The growth of a naled in nature is generally complicated by both unsteady and three-dimensional effects.
Figure 1.1. Schematic of simplified, two-dimensional naled ice growth in the refrigerated flume.
The present study is apparently the first for which naleds were grown and monitored in a laboratory. To aid in the design of the experiments and in the interpretation of the data, a preliminary objective of the study is to produce a synthesis of the literature on naleds in nature. As original contributions to the study of naled ice growth, the primary objectives of the study are the following:

1. To develop a theory of naled ice growth based on physical conservation laws, dimensional analysis, and laboratory observations;
2. To provide a detailed description of the manner in which naled ice grows as a naled spreads and thickens;
3. To collect laboratory data to guide and verify the theoretical formulation and dimensional analysis of naled ice growth.

The theory, descriptions, and data presented apply primarily to the growth of two-dimensional, or laterally confined, naleds. However, the concepts and methods introduced apply also to the growth of naleds that have other geometries.

Outline of Dissertation

In Chapter II, the literature that is pertinent to the present theoretical and experimental study of naleds is reviewed. The significance of naled ice growth to engineers is also discussed.

In Chapter III, the characteristics of naled ice growth in nature are discussed in some detail.

A theory of naled ice growth is proposed in Chapter IV. From the conservation of mass equation, a spread length, termed the equilibrium
length, is derived and used in the identification of appropriate length and time scales for describing naled ice growth. These scales, together with the conservation equations, dimensional analysis, and the laboratory data are used for the identification of the key dimensionless parameters that influence naled ice growth.

The experiments are described in Chapter V, and the observations from the experiments are discussed in Chapter VI. A composite description of the processes associated with naled ice growth is presented.

In Chapters VII and VIII, the data collected from the experiments are presented and compared with the theory introduced in Chapter IV. The data define the effects of each of the key dimensionless parameters that influence naled ice growth.

A brief discussion concerning the application of the experimental results to the growth of naleds in nature constitutes Chapter IX. Chapter X contains the conclusions from the study, and includes recommendations for further research on naled ice growth.
CHAPTER II
REVIEW OF LITERATURE

Overview of the Literature on Naled Ice Growth

The literature on naled ice growth is not extensive and consists largely of anecdotal articles. Common topics of discussion include the spatial distribution, size, and engineering significance of naleds; the assumed hydrological, meteorological, and geomorphological conditions that cause their formation; and methods of controlling and preventing them.

Further, the literature contains neither a detailed theory for naled ice growth, nor any data from laboratory experiments, nor even a detailed description of the manner in which naleds spread and thicken. Some articles vaguely express naled ice growth as being proportional to, or a function of, the ambient natural conditions. Other articles contain empirical relations, based on the analysis of field data, for gross naled properties such as volume and surface area. These relations usually apply only to the localities from which the field data were collected.

Most of the literature on naleds originates from either the Soviet Union, the United States, or Canada, with the majority from the Soviet Union (the present literature search discovered one reference from Poland (Baranowski 1982)). The bibliography in this dissertation
contains a list of all the articles collected during the present literature search, including many that are not directly referenced in the text. Because an annotated bibliography compiled by Carey (1970) lists the pertinent works published before 1968, the emphasis of the present literature search was placed on reviewing articles published after 1968. Carey's bibliography contains brief summaries of 94 selected works concerned with naleds.

Perhaps the most useful publication on naleds is Carey's monograph (1973), in which the then-current knowledge is summarized. This monograph is still very useful because little new knowledge on naleds has appeared in the literature since 1973. Another valuable work is the monograph Siberian Naleds, edited by Alekseyev et al. (1973). This work comprises 23 selected papers by authors from the Soviet Union and contains an extensive bibliography. The books by Muller (1947) and Liverovsky and Morozov (1941) contain extensive discussions on naleds and the problems caused by them. The results of noteworthy field studies are reported by Sloan, Zenone, and Mayo (1976); Thomson (1963); Eager and Pryor (1945); Sokolov (1973a and 1979); Harden, Barnes, and Reimnitz (1977); Tolstikhin and Sokolov (1972); Savko (1973); Gavriloa (1972); Osokin (1973); Johnson and Esch (1977); and Kane (1981).

**Review of Previous Analytical and Empirical Work**

The first attempt to relate the extent of naled development to the conditions that govern naled ice growth was apparently that by Podyakonov (cited by Carey 1973), who in 1903 wrote a formula for the
"intensity," $I$, of naled development over a bed of alluvium in a valley. The Podyakonov formula is

$$I = F \frac{c}{b} Q \frac{W}{a + A}, \quad (2.1)$$

in which $F = \text{the magnitude of the difference between the local air temperature and the freezing temperature of water}$, $c = \text{the heat conductivity of the valley alluvium beneath a naled}$, $b = \text{the thickness of the snow cover}$, $Q = \text{the water discharge that feeds the naled}$, $W = \text{the width of the river valley}$, $a = \text{the cross-sectional area of the unfrozen valley alluvium}$, and $A = \text{the cross-sectional area of the stream channel}$. This formula, at best, merely indicates the assumed relationships between the variables, and is useful only as an illustrative device. The dimensions in (2.1) are not consistent, and the naled intensity is an unclear measure of a naled's size.

After critical examination by many researchers, the Podyakonov formula was modified and presented as

$$I = \frac{F A w X 1}{B Q d T a'}, \quad (2.2)$$

in which $B = \text{the area of the river drainage basin above the reach under study}$, $w = \text{the stream width}$, $d = \text{the stream water depth}$, $X = \text{the depth of frozen soil on the banks of the stream}$, and $T = \text{the water temperature of the river at the location of the naled}$. (Carey 1973). As for the original Podyakonov formula, (2.2) is only useful for qualitatively illustrating the relationships between the variables.
Of primary interest to engineers are the total volume of ice in a naled, the horizontally projected surface area of a naled, and the average depth of a naled. Consequently, much research has been devoted to determination of the functional dependence of these variables on the conditions that govern naled ice growth. In 1941, N. I. Tolstikhin (cited by Bukayev 1973) proposed the following equation for the volume of a naled:

\[ V = kQt \pm C, \quad (2.3) \]

in which \( V \) = the total naled volume, \( Q \) = the source-water discharge, \( t \) = the time over which the naled has developed, \( k \) = a coefficient apparently meant to represent the difference in specific gravity between water and ice, and \( C \) = a coefficient representing the effects of evaporation, precipitation, condensation, and any other process that removes or adds water to the naled but is not included in the source discharge, \( Q \). This formula is a reasonable representation of the principle of conservation of mass for water added to and removed from a naled. While seemingly trivial, (2.3) is the nearest to an analytical relationship for naled ice growth that can be found in the literature.

Some useful empirical formulas for naled ice growth have been developed. For example, Alekseyev and Tolstikhin (1973) present formulas for the specific area and specific volume of naleds (apparently introduced by O. N. Tolstikhin in 1966). The specific area, \( S_s \), is the naled surface area per unit source discharge and the specific volume, \( V_s \), is the naled volume per unit source discharge, or
\[ S_s = \frac{S}{Q} \quad (2.4) \]

and

\[ V_s = \frac{V}{Q} \quad (2.5) \]

in which \( S \) and \( V \) are the nalled surface area and volume, respectively, and \( Q \) is the average source discharge. These relationships are useful because the conditions under which nalleds form are often similar at various locations. Average values of specific area and specific volume can be used to predict the extent of nalled ice growth in the geographical region from which the average values were determined and also to predict, at least approximately, the extent of nalled ice growth in regions of similar climate. Tolstikhin and Sokolov (1972) provide some values of specific areas that apply to conditions that are characteristic for eastern Siberia: for nalleds with surface areas greater than 1 \( \text{km}^2 \), \( S_s \) is about 0.011 \( \text{km}^2/\text{liter/second} \) and for nalleds with surface areas less than 1 \( \text{km}^2 \), \( S_s \) is about 0.008 \( \text{km}^2/\text{liter/second} \). Values for \( V_s \) are not often reported. Note that \( V_s \) has units of time, and represents the time of formation of a nalled fed only by source discharge \( Q \), with no other sources of water.

Sometimes, empirical relations for nalled volume as a function of nalled surface area are included in the literature. Tolstikhin and Sokolov (1972) gave the following relationship, applicable to eastern Siberia, for the nalled volume at the end of winter:
\[ V = 0.96S^{1.09} , \]  

(2.6)

in which \( V \) = naled volume in thousands of cubic meters, and \( S \) = naled surface area in thousands of square meters. Similarly, Osokin (1973) gave the following relations for end-of-winter naled volumes in the Trans-Baikal region of Siberia: for very small, small, and medium naleds

\[ V = 0.9S , \]  

(2.7)

and for large and very large naleds

\[ V = 1.12S . \]  

(2.8)

It must be conceded that empirical relations of these kinds, while crude and not based on physical reasoning, are useful to engineers interested in a rough estimate of the size that a naled can achieve under a given source discharge.

One intriguing article (Savko 1973) contains mathematical relations for the volume, surface area, maximum and average thickness, and length of a naled in terms of values for the variables that govern its growth. Savko claims that the relations are based on an analysis of the literature, field studies of various types, laboratory studies of the freezing of a thin layer of water flowing over ice, observations of the functioning of anti-naled control structures, and solution of the pertinent heat-transfer and hydraulic relations. However, no details are given nor are any references that may contain the details supplied. The rela-
tions that are given appear to be nothing more than curve fitting of field or experimental data. Some of the relations are neither consistent nor even dimensionally correct.

Although little analytical work on naleds is available, the general problem of solidification of a liquid flowing over a heat conducting wall has been treated (e.g., Stephan 1969, Schneider 1980). However, in none of these works is the fluid considered to have a free surface at which heat transfer takes place in addition to that at the wall. Also, in these works, the fluid is never permitted to become supercooled, which inevitably happens to the water flowing over a growing naled. While these works are somewhat useful as examples of the heat transfer processes at a phase-change interface, they are not directly applicable to the present study.

Gilpin (1977, 1981a, 1981b) and Ashton (1984) have conducted studies concerned with the freezing of water in pipes. These studies, on a subject seemingly similar to naled ice growth, are not directly relevant because the heat transfer from a free water surface is not the primary heat flux for freezing in pipe flow.

**Engineering Significance of Naleds**

With the construction of roads, railroads, airfields, bridges, pipelines, and structures in the cold regions, naleds have posed problems of engineering significance. Some of the problems caused by naleds, as well as some of their favorable aspects, are briefly discussed in this section. For further information on problems attributable to naled ice growth the reader is referred to Carey (1970, 1973,
and 1977), Carey et al. (1975), Liverovsky and Morozov (1952), Muller (1947), and Eager and Pryor (1945).

Man-made structures in the path of naleds may be threatened by them. Figure 2.1, for example, illustrates an instance in which a river naled engulfed a small bridge. As is illustrated in Figure 2.1, a naled fed by a relatively small discharge of water can grow to a surprisingly large size. River naleds often spread onto a river's flood plain where they may threaten buildings. An example is illustrated in Figure 2.2, in which the ice surface is above the windowsills of the buildings. At the right end of the building in the center of Figure 2.2, the top of a nearly submerged car is visible. In contrast to a water flood, which lasts a relatively short time, an "ice flood" like this leaves an area uninhabitable until the following summer, when the ice finally melts.

Ground naleds cause persistent maintenance problems on Alaskan and Canadian highways (Eager and Pryor 1945, Thomson 1963), especially along highway cuts, which lead groundwater to seep over roadways. Two examples are shown in Figure 2.3. Figure 2.3a illustrates a situation in which man's engineering activities resulted in the growth of a naled. An embankment constructed across a small drainage course permitted deeper penetration of the seasonal frost than had occurred when the ground was covered by vegetation. As a result, the subsurface flow was blocked, causing the water to emerge at the surface upstream from the blockage and feed the growth of a naled. Figure 2.3b illustrates an anti-naled measure sometimes used: a barrier (not sufficiently high in this case) to contain the naled. Once a road is covered by a naled it
Figure 2.1. Naled on a small river near Fairbanks, Alaska (from Carey 1973).
is usually impassable until the ice has been ripped away by heavy machinery. Naleds often inundate portions of railroads and airfields, in addition to roads.

Naleds often grow in culverts or other drainage structures, sometimes blocking them and, consequently, the drainage routes for subsequent runoff. Culvert blockage can cause flooding and washouts of embankment material in the spring. By filling river basins, even
(a) road blocked by a naled (reproduced from Brown 1970, with permission of University of Toronto Press)

(b) naled near Paxton, Alaska retarded by a "snow-fence" (reproduced from Andersland and Anderson 1978, with permission of McGraw-Hill)

Figure 2.3. Examples of naleds that threaten roads.
spreading to the flood plain, naleds can block river cross sections resulting in spring flooding.

Figure 2.4 illustrates a case in which groundwater invaded a residence and, by freezing, eventually filled it with ice. The presence of a warm building can result in locally unfrozen ground which provides a path to the surface for groundwater that is under pressure.

![Diagram of naled invading a house in Siberia](image)

Figure 2.4. Naled invading a house in Siberia (from Muller 1947).

There are some favorable aspects of naleds. The freely flowing water that feeds naleds sometimes serves as a source of water in winter. Large naleds that occur year after year are good indicators of large and dependable groundwater supplies. Natural and man-made naleds from which water is obtained in the warm parts of the year for industry, for irrigation, and to extend the timber floating season were referred to by Sokolov (1973b) and Are (1973).
Naleds also affect the local geomorphology and hydrology of a region, effects that cannot be classified as necessarily favorable or unfavorable. Naleds re-work valleys by cutting terraces and widening floodplains (Sokolov 1973b). Naleds act as storage reservoirs of water in the winter, slowly releasing the water in late spring and early summer. In some cases, water from thawing naleds is the major component of a river's summer water supply (Doganovskiy 1973).

In response to the difficulties imposed by naleds, men have attempted to devise techniques for counteracting them. Some techniques seek to avoid naleds, some to control them, and some to prevent them from growing. A summary of various methods is given by Carey (1973).
CHAPTER III
NALED ICE GROWTH IN NATURE

In this chapter, naled ice growth is described using published observations on the origin, occurrence, size, and characteristics of naleds. Although frequent reference is made to the literature, no attempt is made to cite every article that has contributed to each of the topics discussed. The chapter concludes with a brief summary of factors that need to be taken into account for designing laboratory experiments on naled ice growth.

The term "naled" is Russian. As mentioned above, naleds are also referred to as aufeis (from German) and icings. We have chosen to use the term naled rather than aufeis partly because naled refers to a body of ice, whereas aufeis refers to a type of ice. One might say that a naled is composed of aufeis. Also, because the Russians have had more experience and problems with naleds than have other peoples, it seems appropriate to use their term. The term "icing" is too generic; besides applying to naleds, icing is used to designate many other phenomena, sometimes referring to things and sometimes to processes. In addition to these designations for naleds, naleds that do not melt completely away during the summer are sometimes referred to as taryns (from Russian), although in recent years this term has come to apply only to ground naleds (Carey 1973).
Naileds are commonly classified in accordance with their water source. The most common types of naleds are river, spring, and ground naleds, which grow from river water, spring water, and groundwater, respectively. Of these three classes of naleds, river naleds are usually the largest, often reaching 10 to 20 km in length and 1 km in width, with average thicknesses of 1.5 to 3.5 m (Alekseyev et al. 1973).

Naleds begin growing when an insulated flow path under an ice cover or underground is interrupted, causing water to emerge to the surface and flow in frigid air. Two typical situations that can lead to naled ice growth are depicted schematically in Figure 3.1. When an ice cover on a shallow river freezes to the river bottom at a location where the underlying pervious alluvium cannot convey the entire discharge (Figure 3.1a), the flow may become pressurized and seep through cracks in the ice cover and then flow over the ice cover. The flow spreads over and freezes onto the ice cover thereby forming a river naled. When the ground at the base of a hill (Figure 3.1b) freezes sufficiently deep so as to intersect an underlying impervious material, the normal flow of groundwater may be interrupted. The consequent pressure increase causes trapped water to seep to the surface, where it can form a ground naled. Water flow over a naled is typically shallow and laminar.

In cross section, naled ice is typically laminated. This feature is cited in many published descriptions of naled ice. The laminations, or ice layers, are often irregular with respect to structure and color, and are of variable thickness, generally ranging from 1 to 40 cm (Alekseyev et al. 1973, Carey 1973).
Figure 3.1. Conditions that can lead to the development of naleds (adapted from Carey 1973).
A naled in nature is a three-dimensional ice formation. Its growth is governed by a complex, unsteady combination of factors (or independent variables) which include the following: the atmospheric conditions (air temperature, wind, net radiation, evaporation) that affect heat transfer from the surface of the naled to the overlying air; the temperature and thermal conductivity of the underlying base, which govern the heat transfer from the underside of the naled; the topography of the surface over which the naled develops; the discharge, temperature, and spatial distribution of the water that issues onto the surface of the naled; and the amount of snow, or precipitation, that falls onto the naled.

The factors that describe the growth state of a naled (the dependent variables) include its lateral dimensions and thickness, and the discharge, temperature, and depth of liquid water over it. The lateral dimensions of a naled are functions of time; the other dependent variables are functions of both time and position on the naled.

**Occurrence of Naleds**

Examples of naleds can be found most places that have at least moderately cold winters. The most familiar example is perhaps the common icicle that often forms from snowmelt along the edge of rooftops. Examples of the kinds of naleds that can be found in Iowa are included in Figure 3.2. The naled in Figure 3.2a annually grows on a rock cut face bordering the parking lot across the street from the IIHR main laboratory building. This is a ground naled that grows from water that
(a) naled formed on rock face

(a) naled in culvert  (b) "icefall" from a drain pipe

Figure 3.2. Naleds in Iowa.
seeps onto the rock face and freezes there. Figure 3.2b illustrates a naled that developed in a drainage culvert. Figure 3.2c shows a naled fed by water from a drip pipe on the outside wall of the IIHR main laboratory building. All of these naleds share the common characteristic of water freezing onto ice, resulting in a progressive build-up of ice.

Naleds that grow to massive proportions, such that they become engineering problems, require very cold winters to grow and are most often found in the permafrost regions of Siberia, Alaska, and Northern Canada. In these regions, and throughout areas of seasonal frost or permafrost, naleds are a common occurrence. Sokolov (1979) wrote that naleds grow in almost all the rivers of the permafrost zone and in some regions beyond. In an area of Siberia covering 25,300 km$^2$, Nekrasov (1973) counted 125 river naleds with total surface area of 76.6 km$^2$. During a study of naled conditions along 2290 km of the Alaska Highway, Eager and Pryor (1945) counted 221 locations at which naleds had grown. Figure 3.3 indicates the numerous locations at which naleds grow annually along the Sagavanirktok River in Alaska. The study from which Figure 3.3 was taken was primarily concerned with the many naleds that grow along the trans-Alaska pipeline route.

Permafrost is a favorable, but not a necessary, condition for the growth of large naleds. For instance, a large river naled was observed in Montana, which is not a permafrost region, by Stevens (cited by Carey 1973).
Figure 3.3. Naleds along the trans-Alaska pipeline route (from Sloan, Zenone, and Mayo 1976).
Conditions that Influence Naled Ice Growth

Naleds grow when water issues from the ground, spreads out, and freezes. Heat transfer and fluid flow are the dominant physical processes. The natural conditions that influence naled ice growth are related to the local climate, hydrology, geology, and topography. Specifically, air temperature, wind, solar radiation, precipitation, evaporation, condensation, groundwater supply, ground temperature and thermal conductivity, surface slope, and ground cover all can affect the growth of a naled.

The water on a naled's surface freezes due to heat flux from the water to both the air above and to the underlying ice and ground (Anisimova 1973). The sum of the surface and ground heat flux components determines the rate at which the naled thickens. The rate at which the naled thickens affects the extent to which the naled will spread out. For very low air temperatures, the surface water over a naled cannot spread out as far before it freezes as it can for higher air temperatures. Consequently, naleds that grow under very cold conditions are generally thicker and cover less ground area than naleds that grow under moderately cold conditions (Carey 1973, Chistotinova and Tolstikhin 1973, Bukayev 1973).

The heat flux from a naled's surface to the air above is a function of air temperature, wind speed, net radiation, evaporation, and condensation. Because of the interrelationship of these factors, the heat flux rate always increases with decreasing air temperature (Bengtsson 1981).
Naleds can begin to develop at the time of year at which the local mean daily air temperature passes below the freezing temperature of water (0°C). They may continue to grow until either their source water supply is depleted, or the local mean daily air temperature rises above the freezing temperature.

The heat flux from a naled into its underlying ice and ground is a function of the ground temperature and thermal conductivity. The ground surface temperature must be less than or equal to the freezing temperature to support a naled (Harden, Barnes, and Reimnitz 1977). It is difficult to find any information in the literature that concerns the magnitude of the heat flux from a naled into the ground, but it seems to be tacitly assumed that this heat flux is small, perhaps even negligible, compared with the heat flux from the surface water to the air (e.g., Thomson 1963, Gavrilo 1972).

The accumulated volume of source water supplied to a naled determines the naled's size, or total volume. The temperature of the source water also influences a naled's growth. Water issues onto the surface at a temperature above the freezing temperature. Consequently, an open-water region often exists between a naled's water source and the ice of which the naled consists (Savko 1973, Bondarev and Gorbunov 1973, Bukayev 1973).

The amount and temporal distribution of rainfall during the summer determines the water supply available to feed naleds in the winter (Carey 1973).
The effect of snowfall on a naled's growth depends on the time of year during which the snow falls. Snow acts as an insulator of the ground. Heavy snowfalls early in the winter can inhibit the propagation of the seasonal frost into the ground, while heavy snowfalls late in the winter can reduce the rate at which the ground thaws. Consequently, heavy snowfalls early in the winter reduce the period of naled ice growth whereas heavy snowfalls late in winter can lengthen the period of naled ice growth (Carey 1973). By falling on the surface of a naled, snow also adds directly to the naled's mass.

If snow covers the area over which a naled begins to grow, the initial extent of the naled will be larger than it otherwise would have been. The naled water can flow under the snow over large areas, gradually saturating the snow cover. The snow insulates the water from the cold air above so that usually the water will freeze only after the snow cover is completely saturated (Are 1973). By acting as an insulator, ground cover affects naled ice growth similarly to snow cover.

The geology of a region determines the characteristics of its groundwater flow. The quantity and rate of groundwater flow available to feed naleds depends on the permeability of the earth materials. As is evident in Figure 3.1, the depth of the permafrost or other impervious base also influences naled ice growth. An impervious base near to the ground surface favors naled development (Carey 1973).

Naleds tend to grow more in areas of steep topography than in areas of flatter relief for reasons that include the following: in areas of steep topography, groundwater flow occurs under larger hydraulic
gradients, streams are generally shallower, the depths of pervious material under streams are generally less (providing less capacity for flow), and springs are more common (Carey 1973).

**General Characteristics of Naleds**

The classification of naleds in accordance with the origin of their water source into river, ground, and spring naleds is useful because some generalizations can be made about naleds of each particular type. Most naleds are fed primarily by either river water, ground water, or spring water, but many grow from a mixture of these water sources and are, therefore, difficult to classify exactly. These are often described as mixed-type naleds.

**River Naleds**

River naleds grow on the ice covers of rivers fed by groundwater or perennial springs. They can grow on nearly any small stream and on some larger streams that are relatively shallow. They often grow at points along a river at which the river slope is abruptly decreased and the river cross section expands and becomes braided, such as where a river leaves a mountainous region and flows onto a plain (Sokolov 1973b, Alekseyev 1973, Carey 1973).

As was illustrated in Figure 3.1a and discussed above, river naleds begin growing when the cross section of flow at some point along a river becomes mostly frozen, thereby constricting the flow. Another situation in which a river naled can develop is illustrated in Figure 3.4. Under some circumstances, a river's ice cover may thicken until it reaches the river's bed over a relatively long reach. The resulting river naled, if
one occurs, may grow near a transition point between underlying pervious alluvium that can transmit the entire winter water flow and less pervious alluvium that cannot transmit the entire flow. This constriction of the flow can cause the water pressure to be increased until the ice cover cracks and water seeps onto the ice cover. The depth of naled water on the surface of a river ice cover is usually small (from less than 2 cm to as much as 30 cm according to Carey (1973)).

![Diagram showing the layers of the ground and water flow](image)

**Figure 3.4.** A situation that can lead to the development of a river naled (adapted from Carey 1973).

River naleds ordinarily begin growing soon after a river's ice cover has formed, which for northern latitudes occurs sometime during the period from October to December (Carey 1973). Their growth is
sometimes steady, and, if the water supply is interrupted periodically, sometimes intermittent. River naleds typically grow most of their volume in the first half of winter during which much of the available water supply is depleted. If source water remains available, they will continue to grow until the mean daily air temperature rises above 0°C, which is usually sometime in March or April. River naleds usually melt by the end of June but some, termed perennial naleds, persist throughout the summer.

River naleds tend to recur every year in nearly the same location. The areas in which they grow are affected by the regular naled development and are called "naled glades" by Russian researchers. These glades are generally flat and broad areas without trees, with grassy vegetation that is retarded in its development and coated by a whitish incrustation of salts. The river bed over a glade is almost always divided into a series of meandering branches. Even in the summer, when no ice is present, these naled glades can be identified as locations of river naleds (Alekseyev 1973, Balobayev 1973, Nekrasov 1973).

Of the three classes of naleds (river, ground, and spring), river naleds become the largest, sometimes reaching 30 km in length and 1 km in width, with total water volume exceeding 10 million cubic meters. They are often as long as 10 to 20 km. An example of a large river naled that has spread onto a river's flood plain is illustrated in Figure 3.5. Sometimes, in the same winter, naleds grow at many locations along a particular river channel and eventually merge, resulting in a continual chain of naleds perhaps 100 km long (Nekrasov 1973). The
average thickness of large river naleds is usually between 1.5 and 3.5 m, but parts of them are sometimes as thick as 10 m.

Figure 3.5. A river naled in Alaska (from Sloan, Zenone, and Mayo 1976).

Ground Naleds

Ground naleds grow when groundwater seeps to the ground surface in frigid weather and subsequently freezes. They sometimes develop on nearly level ground, sometimes at the base of a slope, and sometimes as encrustations on slopes (Carey 1973). Ground naleds that grow on nearly vertical slopes are often referred to as icefalls.
As was illustrated in Figure 3.1b and discussed above, ground nalesds begin growing when a groundwater flow path is interrupted. The subsequent increase in water pressure causes water to emerge to the surface. Often this water seeps up along the roots of trees or shrubs, follows the tunnels of burrowing animals, or issues from frost induced cracks in the ground (Carey 1973). The advancing seasonal frost which lies above the pervious water bearing material (these unfrozen, water bearing regions of ground are called taliks in the USSR) contributes to the pressurization of the groundwater. As illustrated in Figure 3.6, the seasonal frost can sometimes trap a lens of underground water. In this situation, the advancing frost pressurizes the water, causing it to break through to the surface and subsequently feed a naled's growth.

As mentioned earlier, ground nalesds are often caused by excavation activities in the cold regions. Figure 3.1b can be used to illustrate this point. Here a ground naled has formed because the seasonal frost merged with the underlying impervious material. This situation is often the result of an engineering project, such as a road, in which the natural vegetation and organic soil, which are good insulators, have been removed and replaced by construction materials, with the result that the seasonal frost advances deeper into the ground than previously. Consequently, ground nalesds often grow near and interfere with engineering works in cold regions. Carey (1973) remarked that ground nalesds are relatively rare under natural conditions, but are the predominant naled type in areas at which the natural conditions have been disturbed by man.
Figure 3.6. A situation that can lead to the development of a ground naled (adapted from Carey 1973).

The depth of the water on the surface of a ground naled is usually small, seldom deeper than about 2.5 cm. The water is sometimes turbid and often contains dissolved minerals or organic constituents with the result that the subsequent ice has a yellow-brown color (Carey 1973). Ground naleds usually, but not necessarily, grow from suprapermafrost water (groundwater present above the permafrost).

Because the size of a naled depends primarily on the magnitude of its source water supply, which is usually relatively small for ground naleds, ground naleds are generally small compared with river and spring naleds.
Spring Naleds

Spring naleds grow from spring water. These naleds differ from ground naleds mainly in that, from a spring, a continuous supply of source water issues, with the result that spring naleds develop through the entire winter and can become much larger than typical ground naleds. For example, Carey (1973) described an enormous Siberian spring naled that annually grows to about 26 km in length, 5 to 8 km in width, and up to 4 m in thickness at some points along its extent. Bondarev and Gorbunov (1973) described a spring naled at least 10 km long and several hundred meters wide. A relatively small spring naled is shown in Figure 3.7. The water that feeds spring naleds is often subpermafrost (below the permafrost) in origin.

Figure 3.7. A spring naled in Siberia (from Carey 1973).
Laminations in Naled Ice

Nearly every description in the literature of the appearance of naled ice refers to the ice as being laminated, stratified, or multi-layered; this is true for all nales regardless of the origin of their water supply or the conditions under which they grow. Indeed, in the introductory paper in Siberian Nales (Alekseyev et al. 1973) Alekseyev and Tolstikhin wrote, "The main criterion characterizing the nales is their stratification which results from the layer-by-layer accumulation." It is interesting that many writers refer to naled ice growth as "layer-by-layer" or as the "successive freezing of sheets of water" (Thomson 1963), as if a layer or film of water flows onto the surface, freezes, and then a new film of water flows onto the surface and freezes. This implies that the naled water flow is intermittent. Yet, after describing naled ice growth as by freezing of successive sheets of water, Carey (1973) wrote that the source water flow may be continuous or intermittent, and many writers discuss nales that are supplied by constantly flowing sources of water (e.g., Bukayev 1973, Chistotinova and Tolstikhin 1973). Figure 3.8 illustrates the laminated character of naled ice, as well as demonstrating the thickness that nales can attain. In the paper in which this photograph originally appeared its caption read, "Layering in ice shows that numerous overflows make up the icing," again suggesting that intermittent flow was responsible for the layering. Another interesting photograph of laminated naled ice is included in Nansen's (1897) account of his voyage in the gyre of the polar ice cap. Nansen found the naled ice in the polar ice cap at latitude 85-degrees north.
Figure 3.8. Naled cross section illustrating ice laminations (from Sloan, Zenone, and Mayo 1976).

As reported in various studies, the thickness of the ice layers is variable. Rudavin and Fedorov (1973) reported 40 to 50 ice layers ranging in thickness from 3 to 40 cm. Bukayev (1973) reported layers 15 to 20 cm thick, with the lower layers thicker than the upper layers. Bondarev and Gorbunov (1973) reported 20 layers ranging in thickness from 3 to 10 cm in one naled, and 12 layers ranging in thickness from 1 to 10 cm in another. Chistotinova and Tolstikhin (1973) reported ice layers of thickness ranging from 2 to 15 cm.
The ice layers are often irregular with respect to structure and color as well as to thickness. Solid ice layers have been observed most often, but thin, porous layers of ice crystals in zones that were formerly occupied by liquid water, that has since been drained, are sometimes found (Carey 1973). Also, unfrozen, slushy layers near the surface of a naled have been reported (Carey 1973), as have layers formed from water mixed with snow (Bukayev 1973). Bondarev and Gorbunov (1973) described layers 30 to 35 cm thick consisting of columnar crystals 4 to 5 cm in diameter, and other layers composed of granular ice. They claimed that the former layers formed from pure water while the latter layers formed from water mixed with snow. Naled ice may be clear, or white as a result of entrapped air bubbles. It is occasionally yellowish or tan colored because of dissolved minerals in the source water. Commonly the color is noted as blue, the result of the physical light absorption spectrum for ice. Sand or other particles are sometimes present in the ice (Carey 1973).

The Water from which Naleds Grow

The water flow over growing naleds is almost always laminar. It is difficult to find this fact explicitly mentioned in the literature, but the flow is always defined in terms of "thin films" of water of small depth and discharge rate.

The temperature of the naled water as it issues to the surface from its source is generally reported to be between 0.5 and 5°C. The temperature of the water supplying a spring naled was reported by Nevskiy and
Nekrasov (1973) to vary between 0.6 and 2.6°C as the air temperature varied between -43 and +17°C. From borehole measurements, Nekrasov (1973) found that the water temperature in a particular aquifer was relatively stable over an entire year, varying between 2.8 and 3.2°C. In general, the temperature of naled source water remains nearly constant because it originates (even in the case of river naleds) underground, where it is protected from the elements.

The discharges of source water to naleds are variable. The spring whose water temperature was measured by Nevskiy and Nekrasov was reported to have a discharge of 20 liters/second. Rudavin and Fedorov (1973) reported the source discharges for two different naleds to be 15 to 19 liters/second for one and 10 liters/second for the other. In a study of a gigantic spring naled, Bukayev (1973) reported that the discharge varied between 200 and 1000 liters/second during the winter months. He also reported that 90 to 92% of the total naled volume was due to the spring, 4 to 5% was due to precipitation, and the remaining 2 to 3% was due to condensation. Chistotinova and Tolstikhin (1973) reported a discharge of about 200 liters/second which fed a naled with surface area of about 2.5 km².

The source of a naled's water is not always stationary during the growth of the naled. Frequently the water will surface at several locations simultaneously. Often hummocks, or elliptically shaped ice mounds, form with deep cracks that run along their axes and through which water finds its way to the naled's surface. The long axes of these hummocks are oriented in the direction of the flow. As the naled
thickens, the cracks in the hummocks inevitably freeze and water breaks through to the naled's surface at some other point. In this way the water source may wander about, effectively leveling the naled depth (Are 1973). The source of water to spring naleds that form on a slope is often gradually displaced up the slope in winter causing the naled to grow up the slope as well as down the slope (Liverovsky and Morozov 1952).

Water is not necessarily distributed evenly over the surface of a growing naled. Anisimova (1973) referred to the flow of water over the surface of a naled as erratic. Are (1973) wrote that the naled he was studying never grew simultaneously over the entire valley that it covered.

The Surface Appearance of Naleds

The appearance of a naled's surface depends on whether the naled is actively growing or completely frozen at the time it is observed. In general, a naled's surface is convex in shape, although it can be plane in its mid-region. The surfaces of inactive, or completely frozen, naleds are sometimes reported as smooth and other times reported as rough, interrupted by hummocks and sinkholes (Rudavin and Fedorov 1973). No doubt there is a wide variation in surface features depending on the particular conditions under which a naled grows.

Little has been written about the surface appearance of active naleds, almost as if the appearance is obvious and need not be discussed. This is curious in light of the experimental observations from
the present study of the formation of an ice-water slush on the surface of a naled. There should be many accounts of the slushy appearance of the water flow over a naled, but this characteristic is alluded to only a few times. Anisimova (1973) referred to the rapid growth of crystals within the naled water and the freezing together of the slush ice. Carey (1973) described the surface of river naleds as dry, wet, slushy, or snow covered. He also wrote of the poor traction that a vehicle can expect on a water or slush covered naled, and of vehicles that have become stuck in the slush cover of an actively growing naled.

**Thawing of Naleds**

As already mentioned, the focus of this study is on the growth of naleds rather than on the thawing of naleds. However, in order to complete the general description of naled characteristics, in this section a few aspects of the thawing process are described.

Naleds start to thaw in the spring, usually in April or May, when the mean daily air temperature rises above 0°C. Naleds thaw rather slowly but most are melted by the end of June. They thaw from the action of heat transfer from the air to the naled and from erosion caused by warm water flowing over the naled surface.

Naleds do not usually thaw uniformly over their surface. Water runoff which erodes naleds gradually becomes concentrated into channels, eventually cutting paths through the naled (presumably the photograph in Figure 3.8 was made possible by this type of erosion). Water sometimes flows between a naled and the ground, cutting under-ice channels (Bukayev 1973).
Relevance to Laboratory Experiments

For the laboratory experiments (described in Chapter V), the reported characteristics of nales in nature suggest the following considerations:

1. The discharge of source water to a laboratory naled should be small enough to ensure that the flow over the naled is laminar;

2. The temperature of the water supplied to a laboratory naled should be no more than a few degrees above the freezing temperature;

3. The temperature of the flume floor over which a laboratory naled spreads should be slightly below the freezing temperature, but the heat flux from the naled into the flume floor should be small;

4. The air temperature in the refrigerated room that houses the flume should be cold enough to simulate the winter climate in Alaska and Siberia;

5. A laboratory naled should develop in a layer-by-layer manner, so that the subsequent naled ice becomes laminated.

Subject to the limitations of the laboratory equipment, the design of the experiments was guided by these considerations.
CHAPTER IV
THEORY OF NALED ICE GROWTH

The dependent variables that describe the growth of a laterally confined naled, such as the streamwise spread length or the depth of water or ice-water slush on the naled's surface, depend on at least fourteen independent variables. These include variables that define position on the naled's surface, time, source-water discharge, two heat-flux components, the slope of the frigid base under the naled, and various water and ice properties. Because temperature effects are incorporated into the heat-flux components, only three dimensions (length, mass, and time) are contained in these variables. Therefore, the fourteen independent variables can be combined into eleven dimensionless groups. However, fourteen independent variables can be combined into eleven dimensionless groups in many different ways, and the manner in which the independent variables are combined affects the usefulness of the resulting dimensionless groups, or parameters.

In this chapter, theory applied to two-dimensional naleds is used to determine a useful form for the eleven dimensionless parameters that influence naled ice growth. After showing that one of these eleven parameters may be neglected and three others are essentially constants, the list of eleven is reduced to a list of seven significant parameters. In Chapters VII and VIII, the experimental data are presented in terms of the seven significant independent, dimensionless parameters.
The theory is developed from the principles of conservation of mass, momentum, and thermal energy. Equations that describe two-dimensional flow in the ice-water slush on a naled's surface are derived. The two-dimensional conservation equations are then integrated over the depth of ice-water slush on the surface of a two-dimensional naled to obtain conservation equations that describe one-dimensional flow through the slush. A theoretical spread length for a two-dimensional naled, termed the equilibrium length, is derived by integrating the depth-integrated conservation of mass equation over the length of a naled. The proper form of several of the dimensionless parameters that influence two-dimensional naled ice growth are then identified by non-dimensionalizing the depth-integrated equations, using the equilibrium length as a length scale for normalization. Finally, dimensional analysis guided by the concept of equilibrium length and by the non-dimensional depth-integrated equations leads to the proposal of a list of seven independent, dimensionless parameters that influence the growth of a laterally confined naled which is supplied with a steady source of water discharge under conditions of constant heat flux rates.

Because much of the surface of a growing naled is likely to be covered by ice-water slush rather than by water alone, the theory for naled ice growth is developed under the assumption that slush covers the entire surface. For reference, in Appendix A the conservation equations for single-phase water flow over an ice base are given. These equations apply to flow in the upstream region of a naled's surface, where the temperature of the flowing water has not yet been reduced to the freezing temperature of water.
Energy Budget for Naled Ice Growth

The heat fluxes and temperature profiles associated with water flowing over a simplified, two-dimensional ground naled are depicted in Figure 4.1. A similar figure could be shown for the heat fluxes associated with the growth of a river naled. As water flows on a chilled base in frigid air it gradually cools, by losing heat to both the air above and the base below (heat fluxes $\phi_{wa}$ and $\phi_{wi}$, respectively) until it reaches its freezing temperature ($0^\circ\mathrm{C}$ for pure water). Because the water cannot cool further without ice growth, the additional heat lost from the water as it continues flowing is balanced by latent heat released when vertical, plate-like ice crystals, or platelets, grow from the ice base into the flow. The growth of ice platelets is discussed further in Chapter VI. In addition to the growth of ice platelets within the flow, ice may either accrete or melt on the base, releasing or absorbing latent heat, depending on whether the heat flux through the base, $\phi_i$, is larger than or less than the heat flux from the water to the base, $\phi_{wi}$.

In Figure 4.1, temperature distributions are shown in both the upstream region of flow, where the bulk water temperature, $T_w$, is greater than the freezing temperature, $T_f$, and in the downstream region, where $T_w$ is equal to $T_f$. These distributions differ only within the flow, in which the water temperature varies from $T_f$ at the water-ice interface to $T_s$ at the water surface. Upstream, $T_w$ is greater than $T_f$. As the water flows downstream, $T_w$ is reduced until the water temperature equals $T_f$ over the entire depth of flow.
Figure 4.1. Temperature profiles and heat-flux components for nalled ice growth.
In nature, the air temperature $T_a$ varies with time and with position over a naled. In the laboratory, $T_a$ was held constant during naled ice growth.

The net heat flux from the water to the air, $\phi_{wa}$, is due to the combined effects of shortwave and longwave radiation, conduction and convection, evaporation, and precipitation. Because it is a complicated function of meteorological conditions, $\phi_{wa}$ is usually expressed semi-empirically as a function of the water surface and air temperatures:

$$\phi_{wa}(x,t) = h_{wa}(x,t)[T_s(x,t) - T_a(x,t)] \quad (4.1)$$

in which $h_{wa}$ is a heat-transfer coefficient which depends on meteorological conditions. Ashton (1978) has suggested an average value for $h_{wa}$ of 25 W/m$^2$·°C for open-water conditions. Bengtsson (1982) measured a value of 12 W/m$^2$·°C in a laboratory cold room. During the present study, $h_{wa}$ was found to vary between about 8 and 11 W/m$^2$·°C as the air temperature in the refrigerated room varied between -5 and -15°C (see Appendix B).

The convective heat flux from the water to the water-ice interface, $\phi_{wi}$, can be expressed as

$$\phi_{wi}(x,t) = h_{wi}(x,t)[T_w(x,t) - T_f] \quad (4.2)$$

As in the case of $\phi_{wa}$, the problem of determining $\phi_{wi}$ is actually a problem of determining the heat-transfer coefficient, $h_{wi}$. Empirical formulas are available for evaluating $h_{wi}$ as a function of flow geometry, a Reynolds number, and the Prandtl number (e.g., see Haynes and
Ashton 1979). In water that is warmer than \( T_f \), \( \phi_{\omega i} \) acts to reduce water temperature.

As is indicated in Figure 4.1, the temperature of the ice beneath the water-ice interface is less than or equal to \( T_f \). Consequently, under the assumption that \( \phi_i \) is conductive, the heat transfer from the water-ice interface into the ice (of thermal conductivity, \( k_i \)) can be stated as

\[
\phi_i(x,t) = k_i \left. \frac{\partial T}{\partial y} \right|_{y=\eta(x,t)}, \tag{4.3}
\]

in which \( \phi_i \) is assumed to be directed vertically downward (for small longitudinal slopes). For the laboratory naleds, illustrated in Figure 1.1, \( \phi_i \) is expressed as

\[
\phi_i(x,t) = \frac{T_f - T_b}{\eta(x,t)/k_i + 1/h_b}, \tag{4.4}
\]

in which \( h_b \) is the convective heat-transfer coefficient for heat flux into the flume floor (\( h_b = 225 \text{ W/m}^2\cdot{}^\circ\text{C} \), Ashton 1971 and Hsu 1973) and \( T_b \) is the temperature of the coolant fluid that circulates under the flume floor. It is assumed for (4.4) that the entire thickness \( \eta \) is composed of solid ice. The change of \( \eta \) with respect to time is sufficiently slow to neglect the heat capacity of the naled ice and initial ice base, and to assume quasi-steady heat conduction.

As is pointed out in Chapter III, for naleds in nature \( \phi_i \) is small, perhaps even negligible, compared to \( \phi_{\omega a} \). In fact, as is illustrated in
Figure 3.1a, river naleds often overlie unfrozen water and, consequently, their temperature is $T_f$ throughout such that $\phi_1$ is zero. Also, as is discussed in Chapter VI, a naled may have layers of slush (at temperature $T_f$) sandwiched between solid layers of ice, with the result that much of the naled is isothermal at $T_f$ and, for the surface layer of slush, $\phi_1$ is zero.

The elevation of the water-ice interface, $\eta$, varies as ice either accretes or melts, releasing or absorbing heat, to balance the difference between the heat fluxes $\phi_{wi}$ and $\phi_1$. The temperature of the water-ice interface is always $T_f$. If the density of ice is designated as $\rho_1$ and the latent heat of fusion for water as $L$, the interface energy budget can be expressed as an equation for the rate of change of $\eta$:

$$\frac{\partial \eta}{\partial t}(x,t) = \frac{\phi_1(x,t) - \phi_{wi}(x,t)}{\rho_1 L}.$$ (4.5)

It is evident from (4.5) that if $\phi_1$ is greater than $\phi_{wi}$, ice will accrete on the interface. Conversely, if $\phi_1$ is less than $\phi_{wi}$, ice will melt at the interface. If the surface flow is slushy ($T_w = T_f$), $\phi_{wi}$ will be zero, and the interface elevation will change in accordance with $\phi_1$. If, as mentioned above, $\phi_1$ is also zero then the interface elevation will be constant and the naled ice will be isothermal at $T_f$.

Ice crystals in the form of ice platelets grow throughout the surface flow through slush ice, as depicted in Figure 4.1. The balance between the heat flux from the naled surface to the air, $\phi_{wa}$, and the latent heat of fusion that is released as the ice platelets grow can be stated as
\[
\frac{\partial V_{si}(x,t)}{\partial t} = \frac{\phi_w a(x,t)}{\rho_i L},
\] (4.6)

in which \( V_{si} \) = the volume of ice platelets per unit surface area of the slush layer exposed to the air. Equation (4.6) may be used to estimate the volume of ice platelets in the naled slush at any position and time.

Consider the region in Figure 4.1 where \( T_w \) exceeds \( T_f \) (upstream from the slushy region). Because \( T_w \) decreases as the water flows downstream, \( \phi_w \) also decreases, as is shown in (4.2). If \( \phi \) is considered to be essentially constant then (4.5) indicates that the rate of ice accretion on the interface increases with increasing \( x \), so the water-ice interface develops an adverse slope which becomes increasingly adverse with the passage of time. Calculation of \( \eta(x,t) \) and \( T_w(x,t) \) in this region requires the simultaneous solution of the conservation of mass, momentum, and thermal energy equations for single-phase flow of water, and (4.5) for the interface elevation, subject to (4.1) and (4.3) as boundary conditions (the conservation equations are given in Appendix A).

Equations for Flow through a Porous Medium of Ice

In this section, the principles of conservation of mass, momentum, and thermal energy are applied to two-dimensional flow through a porous medium composed of ice crystals. The porous medium of ice is assumed to be incompressible. The porosity of the medium, \( m \), is defined to be the ratio of the volume of water (or voids) in the medium to the total
volume of the medium. The porosity of the porous medium can change with
time, as the flowing water freezes or the ice crystals in the slush
melt. Relative to the coordinate system, the ice crystals in the porous
medium are assumed to be fixed in position.

In addition to porosity, areal concentrations of pores, \( m_x \) and \( m_y \),
are defined. These areal concentrations indicate the fractions of the
cross-sectional areas normal to the x and y directions through which
water may flow. In this analysis, the x-direction is considered to be
the horizontal direction and the y-direction is considered to be the
vertical direction.

Conservation of Mass

The components of mass flux into and out of a volume, \( \Delta x \Delta y \), of
porous medium are indicated in Figure 4.2. The volume of porous medium
shown is considered to be small enough so that the outflow flux compo-
nents at \( x_1 + \Delta x \) and \( y_1 + \Delta y \) may be written as Taylor series expansions, in
which only the first-order terms are retained, of the inflow flux compo-
nents at \( x_1 \) and \( y_1 \). The volume of porous medium shown in Figure 4.2 is
considered to be large enough so that the average properties of the
porous medium are represented in the volume. The horizontal and verti-
cal velocity components, \( u \) and \( v \), are seepage velocities, or average
pore velocities, defined such that the mass flux into a differential
volume is as given in Figure 4.2.

By the principle of conservation of mass, over any time period the
net increase of mass within a control volume is equal to the net influx
of mass into the control volume during the time period. At any given
Figure 4.2. Mass flow into and out of a small volume of porous ice.

At any time, the control volume depicted in Figure 4.2 contains a mass of water equal to \( \rho_w m \Delta x \Delta y \) and a mass of ice equal to \( \rho_i (1-m) \Delta x \Delta y \). Water is converted into ice within the control volume at the rate \( \Delta x \Delta y \partial m / \partial t \). The principle of mass conservation yields

\[
\frac{\partial}{\partial t} [ \rho_w m \Delta x \Delta y + \rho_i (1-m) \Delta x \Delta y ] = - \frac{\partial}{\partial x} (\rho_w u_m \Delta y) \Delta x - \frac{\partial}{\partial y} (\rho_w v_m \Delta x) \Delta y,
\]

(4.7)

which can be reduced to

\[
\frac{\partial (m_x u)}{\partial x} + \frac{\partial (m_y v)}{\partial y} + (1 - \frac{\rho_i}{\rho_w}) \frac{\partial m}{\partial t} = 0.
\]

(4.8)
The term \( \dot{m}/\dot{t} \) accounts for the volume of water removed from the flow to grow ice. The term \( (p_i/p_w) \dot{m}/\dot{t} \) accounts for the water that is forced out of the control volume because of the difference in volume between the ice that has accumulated and the water from which it grew.

**Conservation of Momentum**

The principle of conservation of momentum equates the resultant force acting on a control volume to the time rate of increase of momentum within the control volume plus the net efflux of momentum from the control volume. The \( x \)-components of the momentum fluxes into and out of a small volume of porous medium and the \( x \)-components of the forces acting on the volume are illustrated in Figure 4.3. The porous medium resists the water motion, exerting a force per unit weight of fluid that is designated as \( R_x \). The pressure, designated as \( p \), acts on both the water and on the anchored porous medium. The total \( x \)-directed momentum in the control volume at any given time is \( \rho_w \dot{u} \Delta x \Delta y \). If the principle of momentum conservation is applied and the result divided through by \( \rho_w \Delta x \Delta y \), the following equation can be derived:

\[
\frac{\partial (\rho \dot{u})}{\partial t} + \frac{\partial (\rho \dot{u}^2)}{\partial x} + \frac{\partial (\rho \dot{u} v)}{\partial y} = -\frac{1}{\rho_w} \frac{\partial (\rho \dot{p})}{\partial x} + g \dot{m} R_x. \tag{4.9}
\]

A similar application of the principle of momentum conservation to the components of momentum and force that act in the vertical, or \( y \), direction yields
Figure 4.3. Horizontal components of momentum and force acting through a small volume of porous ice.
\[
\frac{\partial (mv)}{\partial t} + \frac{\partial (muv)}{\partial x} + \frac{\partial (m v^2)}{\partial y} = -\frac{1}{\rho_w} \frac{\partial (m p)}{\partial y} + \rho g \frac{R_y}{\rho g} - gm. \tag{4.10}
\]

The third term on the right-hand side of this expression accounts for the force of gravity on the flowing water. The resistance forces, \( R_x \) and \( R_y \), are presumably functions of the porosity of the porous medium, the shape of the ice crystals of which the porous medium consists, and a Reynolds number associated with the flow. Values for \( R_x \) and \( R_y \) must come from experiments.

**Conservation of Thermal Energy**

The temperature of the water that flows through a porous medium composed of ice can reasonably be assumed constant and equal to the freezing temperature of water. Because heat is not transferred through water at constant temperature, the heat fluxes from the boundaries of the control volume, those that balance the latent heat of fusion released when ice grows in the porous medium, are presumably conducted through the ice crystals that compose the porous medium.

Figure 4.4 depicts a control volume of porous medium with boundary heat flux components, \( \phi_x \) and \( \phi_y \). By the principle of conservation of thermal energy, the net efflux of heat from the control volume must balance the latent heat released from ice that grows within the control volume, or

\[
\rho_i \frac{\partial}{\partial t} [(1-m) \Delta x \Delta y] = \frac{\partial}{\partial y} (\phi_y \Delta x) \Delta y + \frac{\partial}{\partial x} (\phi_x \Delta y) \Delta x. \tag{4.11}
\]
The rate of ice growth is assumed to be sufficiently small that one can neglect the heat capacity of the ice in the porous medium, and assume quasi-steady heat transfer. Division of (4.11) by $-\Delta x \Delta y$ yields

$$
\rho_i L \frac{\partial m}{\partial t} = -\frac{\partial \phi_y}{\partial y} - \frac{\partial \phi_x}{\partial x}, \quad (4.12)
$$

which is an equation for the time rate of change of the porosity of the porous medium.

Figure 4.4. Heat flux from a small volume of porous ice.
Depth-Integrated Equations for
Flow Through Naled Slush

If it is assumed that the ice-water slush on a naled may be modeled as a porous medium composed of ice, then the differential equations (4.8, 4.9, 4.10, and 4.12) that represent the conservation laws for flow through a porous medium of ice may be integrated over the depth of the slush on a two-dimensional naled. The resulting equations are applicable to flow through naled slush during the time when the slush porosity is decreasing and the water surface is rising. Once the surface of the slush begins to freeze solid, which eventually occurs (as is discussed in Chapter VI), the depth-integrated equations are no longer applicable, because the nature of the flow is altered and additional forces come into play. However, these equations should apply to the fresh slush layer that would eventually develop on the preceding slush layer's frozen surface, and to each successive slush layer. Thus, dimensionless parameters determined from the depth-integrated equations should be relevant to the overall growth, by accumulation of successive layers, of a two-dimensional naled.

Figure 4.5 is a sketch of the flow through ice-water slush on the surface of a two-dimensional naled. Throughout the slush, the pressure is assumed to be hydrostatic. The slope of the ice surface over which water flows is assumed to be sufficiently small to consider integration in the vertical, or y, direction to be equivalent to integration over the depth -- that is, in the direction normal to the ice surface. Accordingly, the conservation equations are integrated over y, from the water-ice interface elevation, \( y_b \), to the water surface elevation, \( h \).
Figure 4.5. Definition sketch for integration over slush depth.

The ice-water slush is assumed to be isotropic, meaning that its porosity, m, and its areal concentrations of pores, m_x and m_y, are equivalent. It is further assumed that the porosity, m, is constant over the slush depth for given x and t, except that, within an infinitesimally small distance, m smoothly increases to a value of 1.0 at the water surface. The portion of each depth integral that is associated with this transition of m is assumed to be negligibly small. The porosity is equal to 1.0 at the water surface because it is assumed that the water is rising above the porous medium of ice.

The manner in which naled slush develops (described in Chapter VI) casts some doubt on the validity of these assumptions concerning the slush porosity. However, as first approximations, these assumptions are reasonable, and the depth-integrated equations are significantly simplified by the use of these assumptions.
Conservation of Mass

Under the assumption that \( m = m_x = m_y \), equation (4.8) may be rewritten

\[
\frac{\partial (mu)}{\partial x} + \frac{\partial (mv)}{\partial y} + \left( 1 - \frac{\rho_i}{\rho_w} \right) \frac{\partial m}{\partial t} = 0. \tag{4.13}
\]

By integrating (4.13) over the depth of ice-water slush on a naled's surface and applying Leibnitz's rule for differentiating an integral, the following equation can be derived:

\[
\frac{\partial}{\partial x} \left[ m \int_{y_b}^{y_h} u dy \right] + \left( 1 - \frac{\rho_i}{\rho_w} \right) \int_{y_b}^{y_h} \frac{\partial (md)}{\partial t} dy + \frac{\partial m}{\partial x} \int_{y_b}^{y_h} v dy + \left( 1 - \frac{\rho_i}{\rho_w} \right) \int_{y_b}^{y_h} \frac{\partial m}{\partial t} dy = 0. \tag{4.14}
\]

If the depth-averaged velocity is defined as

\[
\overline{u} = \frac{1}{d} \int_{y_b}^{y_h} u dy, \tag{4.15}
\]

then

\[
q = \overline{u} m d. \tag{4.16}
\]

Herein, an overbar above any term denotes the depth average, as defined for \( u \) in (4.15).

The third term in (4.14) is evaluated by using boundary conditions at \( y = y_b \) and \( y = h \). It has already been mentioned that \( m = 1.0 \) at \( y = h \). Also, the velocity components, \( u \) and \( v \), are subject to kinematic
boundary conditions. These conditions require that no flow cross the interface defined by the water surface, and that the only water crossing the water-ice interface is that quantity of water that freezes onto the ice surface. The kinematic boundary conditions take the following form:

\[ v - u \frac{\partial h}{\partial x} = \frac{\partial h}{\partial t} \quad \text{at } y = h \]  

(4.17)

and

\[ v - u \frac{\partial y_b}{\partial x} = (1 - \frac{\rho_i}{\rho_w}) \frac{\partial y_b}{\partial t} \quad \text{at } y = y_b. \]  

(4.18)

If (4.15) through (4.18) are invoked together with the condition that \( m = 1.0 \) at \( y = h \), and if \( h \) is replaced by \( y_b + d \), then (4.14) can be reduced to

\[ \frac{\partial q}{\partial x} + \frac{\partial (md)}{\partial t} + \frac{\rho_i}{\rho_w} \frac{\partial y_b}{\partial t} + \frac{\partial d}{\partial t} - \frac{\partial (md)}{\partial t} = 0. \]  

(4.19)

In (4.19), the expression inside the brackets may be simplified.

For water flow through the ice-water slush that accumulates on a naled's surface, (4.5) with \( \phi_w = 0 \) may be used to replace \( \frac{\partial y_b}{\partial t} \), which is equivalent to \( \frac{\partial h}{\partial t} \). Also, (4.29), which is derived below from the conservation of thermal energy principle, may be substituted for \( \frac{\partial (md)}{\partial t} \).

The resulting depth-integrated equation of mass conservation is

\[ \frac{\partial q}{\partial x} + \frac{\partial (md)}{\partial t} + \frac{\phi_w + \phi_i}{\rho_w L} = 0. \]  

(4.20)
This equation reflects the fact that in frigid air, the water discharge, \( q \), over a two-dimensional naled diminishes with distance along the naled's surface because water is continually removed from the flow to grow ice (third term in 4.20), and because water ponds (second term) when ice growth in the slushy surface flow becomes an increasing impedance to the flow. The effects of snowfall, evaporation, and condensation have been neglected for (4.20). These effects may sometimes be significant for natural naleds.

**Conservation of Momentum**

Under the assumption that \( m = m_x = m_y \), (4.9) may be rewritten as

\[
\frac{\partial (mu)}{\partial t} + \frac{\partial (mu^2)}{\partial x} + \frac{\partial (muv)}{\partial y} = - \frac{1}{\rho_w} \frac{\partial (mp)}{\partial x} + g m R_x. \tag{4.21}
\]

Because the water pressure may be taken as hydrostatic, (4.10), which represents conservation of vertical, or \( y \), momentum, is replaced by

\[
p = \rho_w g (h - y). \tag{4.22}
\]

The following depth-averaged quantities are used for the integrated form of (4.21):

\[
\bar{u}^2 = \frac{1}{d} \int u^2 dy \equiv \beta \bar{u}^2 \tag{4.23}
\]

and
\[ R_x = \frac{1}{d} \int_{y_b}^{h} R_x \, dy, \]  

(4.24)

in which \( \beta = \text{momentum correction factor} \) (\( \beta \) is approximately 1.2 for a uniform free-surface laminar flow). For simplification, \( \beta = 1.0 \) is assumed.

Integration of (4.21) over the depth of ice-water slush on a two-dimensional naled yields

\[ \frac{3(\text{dm} \bar{u})}{\partial t} + \frac{3(\text{dm} \bar{u}^2)}{\partial x} = - \frac{g}{2} \frac{3(\text{md}^2)}{\partial x} + g \text{mdR}_x \]

\[ + \left[ \mu \frac{\partial y}{\partial t} + \mu u^2 \frac{\partial y}{\partial x} - \mu uv + g \text{m}(h-y)\frac{\partial y}{\partial x} \right]_{y=h}^{y=y_b}, \]

(4.25)

in which (4.15) and (4.22) through (4.24) are used, and \( \beta = 1.0 \). If the first term on the right-hand side of (4.25) is expanded and (4.16) through (4.18) are substituted, (4.25) can be reduced to

\[ \frac{\partial q}{\partial t} + \frac{\partial (q^2/\text{md})}{\partial x} + g \text{md} \frac{\partial h}{\partial x} + \frac{gd^2}{2} \frac{\partial m}{\partial x} - g \text{mdR}_x = 0. \]

(4.26)

This expression states that local and convective accelerations (first two terms) of the water flowing through ice slush on a naled's surface are balanced by various forces that act on the flow. The third and fourth terms in (4.26) account for the force imposed on the flow by the horizontal pressure gradient: the third term represents the magnitude of the pressure gradient, in terms of the water surface slope; the
fourth term represents the change with \( x \), due to the horizontal porosity gradient, of the cross-sectional area over which the water pressure acts. The fifth term accounts for the depth-averaged resistance to the water flow, imposed by the ice platelets in the slush.

Conservation of Thermal Energy

For flow through a porous medium composed of ice, the principle of thermal energy conservation was shown earlier to requires that

\[
\rho \frac{A_m}{\partial t} = - \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x}.
\]  
(4.12)

It is assumed that the longitudinal heat flux in the isothermal, ice-water slush on a naled is zero. Therefore, both \( \phi_x \) and \( \partial \phi_x / \partial x \) are zero. Besides the condition that \( m = 1.0 \) at \( y = h \), two additional boundary conditions are needed for the integration of (4.12):

\[
\phi_y = \phi_{wa} \quad \text{at} \quad y = h
\]  
(4.27)

and

\[
\phi_y = -(1- m) \phi_i \quad \text{at} \quad y = y_b.
\]  
(4.28)

The heat flux, \( \phi_i \), into the underlying ice surface balances the latent heat released from ice growth directly on the ice surface and, also, a portion of the ice growth in the slush. The fraction of \( \phi_i \) that balances latent heat released from ice growth within the slush is proportional to the fraction of the underlying ice surface that is in direct contact with slush ice rather than with water.
With the use of (4.27) and (4.28), integration of (4.12) yields an equation for the rate of change of slush porosity, m:

\[
\frac{\partial (md)}{\partial t} = \frac{\partial d}{\partial t} - \frac{\phi_{wa}}{\rho_1 \lambda}.
\]  

(4.29)

In this form, (4.29) states that a change in water quantity at any position, x, is equal to the change in the slush depth at that position minus the quantity of water that is being transformed to ice at that position.

If \( \phi_{wa} \) is constant, (4.29) can be integrated to obtain an explicit expression for m:

\[
m = 1 - \frac{\phi_{wa}}{\rho_1 \lambda d (t - t_o)},
\]  

(4.30)

in which it is assumed that \( m = 1.0 \) when \( t = t_o \).

**Equilibrium Length of a Naled**

The longitudinal length to which a naled spreads depends primarily on the magnitudes of the source-water discharge and the heat fluxes \( \phi_{wa} \) and \( \phi_i \).

When a flow spreads out over a naled it may eventually reach a length at which the rate of water freezing is equal to the rate that water is supplied. In other words, for this length of spread the source-water discharge is equal to, or in equilibrium with, the rate at which water is freezing. Therefore, this length may be thought of as an equilibrium length for the streamwise spreading of the naled. For a
simplified, two-dimensional naled, this length of spread, which is hereinafter referred to as the equilibrium length, can be derived from the depth-integrated conservation-of-mass equation. The equilibrium length serves as a very useful length scale for normalizing the laboratory data on the spreading and growth of laterally confined naleds, and for non-dimensionalizing the depth-integrated equations.

If the rate of water that is freezing along a naled's surface is equal to the rate of water supplied to the naled, then no water should be accumulating within the naled slush. This suggests that $\frac{\partial (md)}{\partial t}$ should be zero. If it is assumed that $\frac{\partial (md)}{\partial t} = 0$, and it is further assumed that the entire surface of a naled is covered by ice-water slush, and that $\phi_{wa}$, $\phi_i$, and $q_o$ are constants, then (4.20) can be integrated from $x = 0$, $q = q_o$ to $x = \ell_e$, $q = 0$ in order to obtain an expression for the equilibrium length, $\ell_e$:

$$\ell_e = \frac{q_o \rho \frac{L}{w}}{\phi_{wa} + \phi_i} \quad (4.31)$$

Equation (4.31) was derived for a two-dimensional naled growing under conditions of steady source-water discharge and heat flux. By applying the conservation-of-mass principle to the two-dimensional surface of a naled, the concept of an equilibrium spread length for two-dimensional naleds can be broadened to define an equilibrium spread area for three-dimensional naleds, over which water is flowing in several directions. However, in nature, naleds probably never grow under steady conditions. In order to define an equilibrium length or an equilibrium
area for naleds in nature, it would be necessary to use space- and time-averaged values of $q_0$, $\dot{\phi}_w$, and $\dot{\phi}_i$.

As they grew, the laboratory naleds developed adjacent solid laminations of ice and alternating layers of solid and slush ice (described in Chapter VI). This process was governed principally by the heat flux from the water surface to the air, $\dot{\phi}_w$. The heat flux through the frigid base, $\dot{\phi}_i$, played a secondary role. The experimental results indicate that the spread length obtained by omitting the heat flux $\dot{\phi}_i$ in (4.31),

$$\ell_s = \frac{q_0 \rho w L}{\dot{\phi}_w},$$

is a more satisfactory length scale than $\ell_e$ for the purpose of describing the distinct phases of naled ice growth observed in the laboratory. For the relatively small values of $\dot{\phi}_i$ likely to occur in nature, $\ell_s$ and $\ell_e$ may be nearly equal; but for the moderate values of $\dot{\phi}_i$ used in the experiments, $\ell_s$ and $\ell_e$ were often significantly different.

The spread lengths $\ell_s$ and $\ell_e$ are physically meaningful approximations of the extent to which a laterally confined naled could develop during different phases of ice growth under given external conditions. For the maximum streamwise extent of a naled during its first phase of growth, before the slush on its surface begins to freeze solid, $\ell_e$ is an appropriate estimate. But, for the maximum streamwise extent of a naled during the later phases of its growth, as it accumulates layer by layer, $\ell_s$ may be a more appropriate estimate, because at times the flow becomes insulated from $\dot{\phi}_i$. However, the spread length attained by a laterally
confined naled will possibly be larger than both \( L_e \) and \( L_s \) for reasons that include the following: in a naled, some water will often flow through slush layers that are insulated from the heat flux \( \phi_w \), the heat flux \( \phi_i \), or both; the surface water may not flow uniformly over all regions of a naled (local dry spots may be present); and, in nature, naleds grow usually from unsteady source-water discharges under unsteady meteorological conditions.

It was observed during the experiments that, for the same discharge of source water and the same flume-coolant temperatures, naleds developing in colder air grew thicker and covered less base area than did naleds developing in moderately cold air. A similar observation has been reported for naleds in nature (Carey 1973, Chistotinova and Tolstikhin 1973, Bukayev 1973). This observation is supported also by (4.31). The equilibrium length, \( L_e \), is inversely proportional to the magnitude of the heat fluxes, so naleds that grow in very cold air have shorter equilibrium lengths than do naleds that grow in moderately cold air. Because at any given time two naleds fed by equal discharges will have equivalent total masses, the naled that is longest will, on average, be also thinnest, and vice versa.

**Dimensionless Parameters Obtained from the Slush Flow Equations**

The correct form and importance of several of the dimensionless parameters that influence naled ice growth can be revealed by properly non-dimensionalizing the equations that represent the conservation laws for flow in the ice slush that typically covers the surface of a growing naled. For reference, the key equations are
\[
\frac{\partial q}{\partial x} + \frac{\partial (md)}{\partial t} + \frac{\phi_{wa} + \phi_i}{\rho_w L} = 0,
\]
(4.20)

\[
\frac{\partial q}{\partial t} + \frac{\partial (q^2/2md)}{\partial x} + gmd \frac{\partial h}{\partial x} + \frac{gd^2}{2} \frac{\partial m}{\partial x} - gmd \frac{\partial r}{\partial x} = 0,
\]
(4.26)

and

\[
\frac{\partial (md)}{\partial t} = \frac{\partial d}{\partial t} - \frac{\phi_{wa}}{\rho_1 L},
\]
(4.29)

in which \( h = y_b + d = y_g + \eta + d \) (see Figure 4.5) and from (4.5) with \( \phi_i = 0 \),

\[
\frac{\partial \eta}{\partial t} = \frac{\phi_i}{\rho_1 L}.
\]
(4.33)

For simplification, it is assumed that the source-water discharge, \( q_o \), and the heat fluxes, \( \phi_{wa} \) and \( \phi_i \), are constant.

If the definition for equilibrium length, (4.31), is used, equation (4.20) can be rewritten as

\[
\frac{\partial q}{\partial x} + \frac{\partial (md)}{\partial t} + \frac{q_o}{\xi_e} = 0.
\]
(4.34)

The dimensional variables \( q, x, \) and \( d \) can be non-dimensionalized as

\[
q^* = \frac{q}{q_o}, \quad x^* = \frac{x}{\xi_e}, \quad \text{and} \quad d^* = \frac{d}{\xi_e},
\]
(4.35)
such that (4.34), after division by $q_o/\ell_e$, becomes

$$\frac{\partial q^*}{\partial x^*} + \frac{\ell_e^2}{q_o} \frac{\partial (md^*)}{\partial t} + 1 = 0. \quad (4.36)$$

The form of the second term in (4.36) suggests that time should be normalized as

$$t^* = \frac{tq_o}{\ell_e} = \frac{t}{100t_e}, \quad (4.37)$$

in which $t_e$ is defined to be equal to $\ell_e^2/100q_o$. The constant 100 is included to make $t/t_e$ of order 1 for the laboratory data. The significance of $t_e$ is that it is the time required to cover a unit width of naled surface, $\ell_e$ in length, with water of average depth $\ell_e/100$. After substitution of (4.37), the non-dimensional form of (4.20) becomes

$$\frac{\partial q^*}{\partial x^*} + \frac{\partial (md^*)}{\partial t^*} + 1 = 0. \quad (4.38)$$

Non-dimensionalization of the momentum equation, (4.26), using (4.35) and (4.37) yields

$$\frac{q_o^2}{g\ell_e^3} \left[ \frac{\partial q^*}{\partial t^*} + \frac{\partial (q^2/2md^*)}{\partial x^*} \right] + md^* \frac{\partial h^*}{\partial x^*} + \frac{d^*}{2} \frac{\partial m^*}{\partial x^*} - md^* \frac{R^*}{x} = 0, \quad (4.39)$$

in which $h^* = h/\ell_e$. The quantity $q_o^2/g$ is equivalent to the cube of the critical depth, $d_{co}$, in a wide rectangular channel which has discharge per unit width equal to $q_o$. 
The slope of the surface over which naled ice growth is initiated can be defined as

\[ S_o = -\frac{\partial y}{\partial x}. \] (4.40)

Then,

\[ \frac{\partial h^*}{\partial x^*} = -S_o + \frac{\partial \eta^*}{\partial x^*} + \frac{\partial d^*}{\partial x^*}, \] (4.41)

in which \( \eta^* = \eta/\ell_e \). Now, (4.39) can be written as

\[ \left( \frac{d}{\ell_e} \right)^3 \left[ \frac{\partial q^*}{\partial t^*} + \frac{\partial (q^*/md^*)}{\partial x^*} \right] + md^* \left( \frac{\partial \eta^*}{\partial x^*} + \frac{\partial d^*}{\partial x^*} - S_o - \bar{R}_x \right) \]

\[ + \frac{d^*}{2} \frac{\partial m^*}{\partial x^*} = 0. \] (4.42)

The dimensionless parameter \( \bar{R}_x \) is analogous to friction slope for computations of open channel flow.

Using (4.32) and the normalized variables defined above, (4.29) is transformed to

\[ \frac{\partial (md^*)}{\partial t^*} = \frac{\partial d^*}{\partial t^*} - \frac{\rho_w}{\rho_i} \frac{\ell_e}{\ell_s}. \] (4.43)

The ratio \( \ell_e/\ell_s \) is equal to the ratio \( \phi_{wa}/(\phi_{wa} + \phi_i) \).

In non-dimensional form, (4.33) is
\[ \frac{\partial h^*}{\partial t^*} = \frac{\rho_w}{\rho_i} \frac{k_e}{k_i} \]  \hspace{1cm} (4.44)

in which

\[ k_i = \frac{q_o \rho_w \frac{L}{\phi_i}}{\phi_i} \]  \hspace{1cm} (4.45)

The ratio \( k_e/k_i \) is equal to the ratio \( \phi_i/(\phi_{wa} + \phi_i) \). Because

\[ \frac{k_e}{k_s} = 1 - \frac{k_e}{k_i} \]  \hspace{1cm} (4.46)

the two ratios \( k_e/k_s \) and \( k_e/k_i \) are not independent of each other.

From the non-dimensional forms of the depth-integrated slush-flow equations, it is evident that the following dimensionless variables and parameters influence naled ice growth:

\[ x^*, t^*, q^*, d^*, \eta^*, m, S_o, \frac{d_{co}}{k_e}, \frac{R}{X}, \frac{\rho_w}{\rho_i}, \frac{\phi_{wa}}{\phi_i} \]  \hspace{1cm} (4.47)

In this list, some variables are independent of the naled ice growth process -- that is, they are determined by initial conditions, fluid or ice properties, or by specification. The independent variables include \( x^*, t^*, S_o, \frac{d_{co}}{k_e}, \frac{\rho_w}{\rho_i}, \) and \( \frac{\phi_{wa}}{\phi_i} \). The dependent variables, those that are dependent on the naled ice growth process, are \( q^*, d^*, \eta^*, m, \) and \( \frac{R}{X} \).
An equation for the resistance parameter, $\overline{R}_x$, has not been included in the analysis, nor has a boundary condition for the downstream extremity of a naled been specified. Consequently, the list of dimensionless parameters, (4.47), is not quite complete. In addition to depending on some of the independent parameters listed above, $\overline{R}_x$ should depend on a Reynolds number for the flow. Furthermore, the experimental results, which will be presented in Chapters VII and VIII, indicated that $\overline{R}_x$ depends on an additional parameter that contains $\phi_{wa}$ in combination with fluid or ice properties. The form of this parameter and the appropriate Reynolds number will be identified in the following section. The boundary condition on the spreading of the downstream extremity, or front, of a naled depends on a combination of the independent variables listed in (4.47) and the additional independent variables that effect $\overline{R}_x$.

Because it is in the acceleration term (the first term) in (4.42), the parameter $d_{co}/k_e$ can probably be omitted from the list of significant independent, dimensionless parameters that affect the growth of a naled. For seeping flow through the ice slush on a naled, it can be expected that the acceleration term in (4.42) is very small. Estimation of the acceleration term, using the experimental data, proved it to be several orders of magnitude smaller than the estimated values of the other terms in (4.42). Therefore, it is satisfactory to rewrite (4.26) as

$$\frac{\partial h}{\partial x} + \frac{d}{2m} \frac{\partial m}{\partial x} - \overline{R}_x = 0.$$  

(4.48)
It is also reasonable to neglect the effect of $d_{co}/L_e$ on the boundary condition at a naled's spreading front. Except, perhaps, for a brief period of time immediately after the growth of a naled is initiated, a naled's front spreads so slowly downstream that fluid acceleration is not significant. In the dynamic sense of accelerations reflecting unsteadiness, the flow through the ice slush on a naled's surface is quasi-steady; that is, at any instant of time the forces acting on the flow are in equilibrium with each other, and the effects of fluid inertia are negligible.

In general, the slope, $S_o$, of the frigid surface over which a naled begins forming represents the effect on naled ice growth of local topography. As defined above, the term $S_o$, which may vary with $x$, applies only to the growth of two-dimensional naleds. For the growth of three-dimensional naleds, the term $S_o$ would be replaced in (4.47) by some parameter, or parameters, that describe a more complicated, three-dimensional topography.

**Dimensionless Parameters that Influence Naled Ice Growth**

Dimensional analysis can be used to complete the list of dimensionless parameters that influence the growth of a laterally confined naled for steady $q_o$, $\phi_{wa}$, and $\phi_i$.

The dimensionless variables and parameters determined from the equations for flow through the ice slush on a two-dimensional naled can also be applied to describe generally the growth of a laterally confined naled. The physical processes that cause ice platelets to grow in the
slush also control the eventual solidification of the slush and the subsequent overflow of the hardened surface with a new layer of ice-water slush.

The dependent variables that describe the state of a laterally confined naled are listed in Table 4.1 and illustrated, for a cross section, in Figure 4.6. Water temperature is not included as a dependent variable because, under the assumption (invoked throughout this chapter) that the water flows through ice-water slush, its temperature is a constant equal to the freezing temperature. Other dependent variables can be defined in addition to those in Table 4.1. For example, the elevations and lengths of the individual ice layers could be included in the list. However, for the purposes of the present study, the list in Table 4.1 is adequate for describing naled ice growth.

The independent variables that influence the growth of a laterally confined naled are listed in Table 4.2. There are no temperatures or heat-transfer coefficients in this list because their effects are represented in the heat fluxes $\phi_{wa}$ and $\phi_i$ (see (4.1) and (4.3)). In addition to the variables that define the growth of a two-dimensional naled, Table 4.2 includes the width, w, of a laterally confined naled. This width is included to account partially for the influence on a naled’s growth of lateral-confining boundaries. As is discussed and shown in Chapters VI, VII, and VIII, the growths of the naleds in a flume were eventually influenced by the presence of the flume’s side-walls. Presumably, if the width of the flume were sufficiently large, a laterally confined naled could be grown that would virtually be two-dimensional during its entire development.


<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Dimensions*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>spread length of a naley</td>
<td>L</td>
</tr>
<tr>
<td>$s$</td>
<td>overall thickness of a naley</td>
<td>L</td>
</tr>
<tr>
<td>$d$</td>
<td>depth of slush on a naley</td>
<td>L</td>
</tr>
<tr>
<td>$m$</td>
<td>porosity of slush on a naley</td>
<td>-</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>thickness of ice laminations</td>
<td>L</td>
</tr>
<tr>
<td>$q$</td>
<td>water discharge per unit width of naley's surface</td>
<td>$L^2/T$</td>
</tr>
</tbody>
</table>

$L = \text{length, } T = \text{time, and } M = \text{mass.}$

If a general dependent variable is designated as $\psi$, the following functional relationship can be written:

$$\psi = f_1(\mathbf{x}, t, q_o, \phi_{wa}, \phi_1, S_o, w, \rho_i, \rho_w, \nu, g, L, \alpha_1, \alpha_w).$$ (4.49)

Here, $\mathbf{x}$ is the vector that defines position on a laterally confined naley's surface. With fourteen independent variables and three dimensions (length, time, and mass) a dimensionless dependent variable, say $\psi^*$, is a function of eleven dimensionless, independent variables. Although there are many ways to combine these independent variables, if the dimensionless variables in (4.47) are used to guide the dimensional analysis, the following functional form can be determined:
Figure 4.6. Dependent variables that describe a naled.

\[ \psi^* = f_2(\frac{x}{l_e}, \frac{t}{t_e}, S_o, \frac{w}{l_e}, \frac{\phi_w}{\phi_i}, \frac{\phi_{wa}}{\phi_i}, \frac{q_o}{v}, \frac{d_{co}}{l_e}, \frac{v}{\rho_i}, \frac{\rho_i}{\rho_w}, \frac{\alpha_i}{\alpha_w}). \] (4.50)

Note that \( l_e, t_e, \) and \( d_{co} \) are algebraic combinations of variables included in (4.49). From (4.50), the three parameters that are ratios of water and ice properties can be omitted because the values of these ratios are essentially constant for ice and water at a temperature near the freezing temperature. Also, for the reasons given above, the term \( d_{co}/l_e \) can probably be neglected. The functional relationship that
Table 4.2
Independent Variables Influencing the Growth of a Laterally Confined Naled

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>vector that defines position along naled's surface</td>
<td>L</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
<td>T</td>
</tr>
<tr>
<td>q_o</td>
<td>source-water discharge per unit width of naled</td>
<td>L^2/T</td>
</tr>
<tr>
<td>( \phi_{wa} )</td>
<td>heat flux from naled surface to air</td>
<td>M/T^3</td>
</tr>
<tr>
<td>( \phi_i )</td>
<td>heat flux from surface water into frigid base</td>
<td>M/T^3</td>
</tr>
<tr>
<td>S_o</td>
<td>surface slope of frigid base under naled</td>
<td>-</td>
</tr>
<tr>
<td>w</td>
<td>width between lateral boundaries of naled</td>
<td>L</td>
</tr>
<tr>
<td>( \rho_i )</td>
<td>density of ice</td>
<td>M/L^3</td>
</tr>
<tr>
<td>( \rho_w )</td>
<td>density of water</td>
<td>M/L^3</td>
</tr>
<tr>
<td>v</td>
<td>kinematic viscosity of water</td>
<td>L^2/T</td>
</tr>
<tr>
<td>g</td>
<td>acceleration of gravity</td>
<td>L/T^2</td>
</tr>
<tr>
<td>L</td>
<td>latent heat of fusion for water</td>
<td>L^2/T^2</td>
</tr>
<tr>
<td>( \alpha_i )</td>
<td>thermal diffusivity of ice</td>
<td>L^2/T</td>
</tr>
<tr>
<td>( \alpha_w )</td>
<td>thermal diffusivity of water</td>
<td>L^2/T</td>
</tr>
</tbody>
</table>

* L = length, T = time, and M = mass.

contains the most significant independent variables that influence the growth of a laterally confined naled, under steady \( q_o \), \( \phi_{wa} \), and \( \phi_i \), is
\[ \psi^* = f_3 \left( \frac{x}{l_e}, \frac{t}{t_e}, S_o, \phi_w^*, \phi_r, \frac{w}{l_e}, R_e \right), \quad (4.51) \]

in which

\[ \phi_r = \frac{\phi_{wa}}{\phi_w + \phi_i}, \quad (4.52) \]

\[ \phi_{wa}^* = \frac{\phi_{wa} (10^{10})}{\rho_w l_e^{3/2}}, \quad (4.53) \]

and

\[ R_e = \frac{q_o}{v}. \quad (4.54) \]

Because \( \phi_{wa}/\rho_w l_e^{3/2} \) is very small, the constant \( 10^{10} \) is included in (4.53) for convenience. The term \( R_e \) is a Reynolds number. The general dependent variable \( \psi^* \) can be replaced by any suitably normalized variable listed in Table 4.1. The dependent variables listed in Table 4.1 are all normalized by \( l_e \), except for \( q_o \) which is normalized by \( q_o \).

For laterally confined naleds that grow under unsteady conditions, (4.51) may require some modification. One possible approach would be to use time and space averaged values for the variables that vary with time and position. Alternatively, the initial values of \( q_o \), \( \phi_{wa} \), and \( \phi_i \) could be used in (4.51), and additional parameters could be included to account for the subsequent unsteadiness of these variables.
The utility of the dimensionless parameters in (4.51) are established in Chapters VII and VIII, where the parameters are used to explain the experimental results. It is shown that the early phase of nased ice growth is influenced by only four of the seven dimensionless parameters in (4.51), and that the later phases of growth depend on six of the seven parameters.
CHAPTER V
EXPERIMENTS

Naleds were grown along the floor of a horizontal or slightly sloped flume under conditions of steady source-water discharge and steady air and flume-floor temperatures. As a naled grew and spread downstream along the flume, the profiles of the water or slush surface on the naled and the underlying ice surface were recorded.

Apparatus

The naleds were grown in a 60-cm wide, 30-cm deep, and 12-meter long refrigerated, tilting flume in the Low Temperature Flow Facility of the IIHR. Details of the flume are shown in Figure 5.1. The side-walls and floor of the flume are constructed of refrigeration panels through which ethylene glycol coolant fluid can be circulated. The temperature of the coolant can be maintained at any value down to about -15°C. Because, in nature, the heat transfer from naleds occurs principally in the vertical direction, for the present study the side-walls of the flume were lined with 5-cm thick insulation panels to minimize heat transfer from the naleds to the side-walls. The flume slope can be varied from 0 to about 2.2%. The air temperature in the refrigerated room can be reduced, by two thermostatically controlled evaporator units, to nearly -20°C, depending on outside temperature and humidity. The
Figure 5.1. Details of refrigerated, tilting flume.
refrigerated flume and the refrigerated room that houses it are more fully described by Kennedy (1970).

The experiments required steady water discharges as small as 0.002 liters/second. These discharges were obtained using the flume's auxiliary pump line to deliver water from the tailbox to a small constant-head tank suspended from the ceiling at the upstream end of the flume (see Figure 5.1). From here the water was fed to a diffuser from which it was discharged uniformly across the flume. Before being discharged into the constant-head tank, the water was passed through a filter to remove small particles that could block the small valve that regulated the flow between the tank and the diffuser. A gate at the upstream end of the flume was closed and sealed so that thenaled water could flow only in the downstream direction.

The temperature of the source water discharged into the diffuser was maintained between about 1 and 3°C so that the water temperature in the pool under the diffuser could be kept between 0 and 0.5°C. The source water was cooled by adding ice to the tailbox, and by closing the bypass valve in the headbox so that water not fed to the constant-head tank was circulated through the cold return line back to the tailbox. As needed, the source water was warmed by either adding warm water to the tailbox, activating the heat tapes wrapped around the supply line, or opening the bypass valve in the headbox so that the recirculated water flowed into the main water supply line, much of which is outside of the refrigerated room.
Controlled Variables

Corresponding to the factors that influence nased ice growth in nature, the variables that were controlled in the laboratory included the source-water discharge, $Q_o$ (or $q_o$ per unit flume width), and source-water temperature; the air temperature, $T_a$; the circulating coolant, or flume floor, temperature, $T_b$; and the flume slope, $S_o$. Table 5.1 is a list of the experiments that were conducted. The discharge of source water, $Q_o$, was set to either 0.002, 0.003, or 0.004 liters/second; $T_a$ was set to either -5, -10, or $-15^\circ$C; $T_b$ was set to either 0, -1, or $-2^\circ$C; and $S_o$ was set to 0 or 0.01. Tests 4b, 22a, and 24a were conducted to duplicate tests 4, 22, and 24, respectively. For each test, Table 5.2 gives the values of the length and time scales for nased ice growth and the values of several of the independent, dimensionless parameters that are listed in equation (4.51).

Most of the experiments were concluded when the nased's front reached the downstream end of the flume. Therefore, it was necessary that the nales spread relatively slowly down the flume in order to have time to collect significant data on the ice growth and spreading. Satisfaction of this criterion required that the nales have equilibrium spread lengths, $l_e$, that are the same order of magnitude as the flume length. To accomplish this, small source-water discharges were required. Also, for many of the tests it was necessary to set the flume-floor temperature to a value that caused the base heat flux, $\phi_i$, to be larger than it would usually be in nature.
### Table 5.1

**List of Experiments**

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Note: The thickness of the initial ice base over which the naleds grew was ~ 0.03 m for tests 1-31 and 0.06 m for tests 32-33.
## Table 5.2
Scales and Parameters

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<th>$t_s$, hrs</th>
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<th>$Re$</th>
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<td>33.6</td>
<td>34.9</td>
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<td>0.21</td>
<td>2.06</td>
<td>2.23</td>
<td>0.071</td>
</tr>
</tbody>
</table>

Note: $t_s = t_s^2 / 100q_o$. 
Source-Water Discharge and Temperature

For most of the tests, the source-water discharge, measured volumetrically, varied no more than ±2% during the course of a test. For a period of time, usually either 30 or 45 seconds, the water flow upstream from the diffuser was diverted into a beaker. A balance with a resolution of 0.1 grams was used to measure the mass of water collected. The collecting time was measured using a stopwatch graduated to 0.1 second.

The temperature of the source water, which varied between about 1 and 3°C during the tests, influenced the size of the water pool beneath the diffuser for distributing the source water. During the experiments, the length of this pool varied up to about 0.3 meters. For each naled, the section x = 0 was taken to be the position at which the water pool ended and the ice-water slush began. With this assumption, varying temperature of source water did not significantly affect the experimental data.

Air Temperature

It is evident from equation (4.1) that the air temperature, \( T_a \), and the heat-transfer coefficient, \( h_{wa} \), determine the heat flux, \( \dot{q}_{wa} \), from a naled's wet surface to the air. As mentioned in Chapter IV, from separate tests (described in Appendix B) \( h_{wa} \) was found to vary between about 8 and 11 W/m\(^2\)-°C as the air temperature in the refrigerated room varied between -5 and -15°C.

The air temperatures listed in Table 5.1 are the mean temperatures measured during each test. The cold room air temperature fluctuated
within an approximate range of \( \pm 1.5^\circ C \) about the mean, as the evaporator units cycled on and off. To maintain the air temperature near \(-5^\circ C\), only one evaporator unit was required. When this unit would shut down to defrost, the air temperature would sometimes warm up to \(-1.5^\circ C\) before the unit cycled back on. The minimum temperature that would occur was about \(-7^\circ C\). To maintain the room air temperature below about \(-7^\circ C\), both evaporator units were required. When one or both evaporator units shut down to defrost, the air temperature would sometimes warm to as much as \(5^\circ C\) above the mean temperature. The minimum temperatures that would occur were about \(3^\circ C\) colder than the mean temperature.

Air temperature in the refrigerated room was monitored using two platinum resistance thermometers, hung from the ceiling over the flume at points \(1/4\) and \(3/4\) of the flume length from the upstream end. These thermometers were connected to a programmable scanner and a printer such that room temperatures could be recorded at any desired interval of time. For most of the tests the recording interval was chosen to be either 15 or 20 minutes -- sufficiently small to obtain a reliable value for mean temperature.

**Flume-Floor Coolant Temperature**

For simplicity, it was desired to maintain the heat flux \( \phi_i \), from a naled into its underlying ice base, constant during each experiment. However, besides being a function (equation 5.4) of the coolant temperature, \( T_b \), and the heat-transfer coefficient, \( h_b \), \( \phi_i \) was also a function of the thickness of the naled. The thickness of each naled,
and thus $\phi_i$, varied with both time and with streamwise position along the naled's length. As a naled thickened in the flume, $\phi_i$ decreased. Also, because the naleds spread slowly down the flume rather than growing uniformly over the length of the flume, at any given time a naled was thicker upstream than it was downstream. Correspondingly, $\phi_i$ was smaller upstream than downstream.

The problem of variations in $\phi_i$ was minimized by growing, on the flume floor, an initial thickness of ice, over which a naled was eventually grown. The thicker the initial ice base, the less that the growth of the naled affected the value of $\phi_i$. The thickness of the initial ice base was about 3 cm for tests 1 through 31. For tests 32 and 33, which were conducted to check the effect on the data of a thicker ice base and corresponding smaller variations in $\phi_i$, the thickness of the initial ice base was about 6 cm. Tests 32 and 33 were essentially duplicates of tests 8 and 2, respectively, except for the differences in the initial thickness of the ice base. As is shown in Chapters VII and VIII, the data collected from tests 32 and 33 were very similar to the data collected from tests 8 and 2, indicating that the experimental data were not significantly influenced by the variations in $\phi_i$.

The values of $\phi_i$, designated as $\phi_{io}$, that are included in Tables 5.1 and 5.2 are those based on the initial thicknesses of ice over which the naleds grew. Similarly, $l_{eo}$ and $t_{eo}$ are the values of $l_e$ and $t_e$ that are obtained by using $\phi_{io}$, rather than $\phi_i$, in (4.31).

The temperature, $T_b$, of the coolant fluid circulated through the flume's floor fluctuated approximately ±0.6°C, because the compressor that cooled the coolant cycled on and off. For the desired mean temper-
atures of 0, -1, and -2° C, a fluctuation of 0.6° C represented a proportionately large deviation in $\phi_i$, which depends on the temperature difference $T_b - T_f$. Furthermore, $T_b$ was measured by means of mercury thermometers embedded at the inlet and outlet ends of the coolant supply lines. At best, the thermometers could be read to a precision of 0.1° C, and a relatively small number of measurements were manually recorded during a test.

Also, because the flume floor was reconstructed during the mid-1970's, the value for $h_b$ obtained by Ashton (1971) and Hsu (1973) was not necessarily applicable for the present flume floor.

The heat flux $\phi_i$ was determined using (4.4), which requires that $T_b$ and $h_b$ be known. The imprecision in the mean value of $T_b$ for each test precluded experiments to determine an accurate value for $h_b$. Rather, as a practical expedient, the following approach was taken: At various times during each experiment, measured thicknesses of the ice base were compared with calculated values of the thickness. The calculations were based on $h_b = 225 \text{ W/m}^2 \cdot \text{°C}$ and an assumed value of $T_b$, which was adjusted until satisfactory agreement between the measurements and the calculations was obtained. This adjusted value of $T_b$, which was generally within 0.5° C of the measured mean value, was used to determine $\phi_{10}$.

Flume Slope

In order to reduce, by one, the number of independent variables (equation 5.51) that influenced the growth of the naleds, the flume slope was set to zero (horizontal) for most of the tests. To obtain
some indications of the effect of slope on a naled's growth, the flume slope was set to 0.01 for tests 28 through 31.

**Procedures**

The first step in conducting an experiment was to grow a level ice base on the floor of the flume. To do this, the air temperature in the refrigerated room was set slightly above 0°C, the temperature of the flume floor was set well below 0°C (by circulating coolant fluid), the flume slope was set to zero (horizontal), and water was circulated through the flume. The heat flux through the flume floor caused a smooth, 3-cm thick ice base to form in about 4 to 12 hours, depending on the coolant temperature. Then the flow was stopped and water was drained from the ice base, leaving it exposed to the air. The upstream gate was closed and sealed, and the air and flume-floor (coolant) temperatures were adjusted to the requisite values for the forthcoming experiment.

An experiment was started when the air and floor temperatures attained the desired values. The flume slope was adjusted (if necessary), the source-water discharge was set and released through the diffuser over the upstream end of the flume, and a naled developed over the upstream portion of the ice base on the flume floor. The progress of the naled as it spread downstream was monitored by recording the times at which the naled's front reached various positions along the flume. Air temperature, coolant temperature, source-water discharge and temperature, and the naled's dimensions were measured regularly as the naled grew. A point gauge, graduated in 0.305-mm (0.01-foot) intervals,
mounted on the flume's instrument carriage was used to measure the water surface and underlying ice surface elevations at various positions along the longitudinal centerline of the naled. The experiment was terminated either when the naled reached the downstream (free overfall) end of the flume, or when the upstream depth of the naled approached the depth of the flume, whichever condition occurred first. The time required for one experiment, from the first release of source water to the end of a test, varied from about 3 to 72 hours.

At the end of each test, the naled was cut to expose and measure the layers of solid and slush ice that characterize cross sections of naled ice.
CHAPTER VI
LABORATORY OBSERVATIONS OF NALED ICE GROWTH

The naleds spread and thickened in a complex, layer-by-layer manner. The first phase of their growth was approximately two-dimensional, but their later growth was complicated somewhat by boundary effects due to the side-walls of the flume.

Water cooled rapidly to its freezing temperature as it flowed over, and fed, a naled. Ice crystals, in the form of platelets anchored to the ice surface, grew into the flow. The accumulation and growth of the ice platelets transformed the free-surface laminar flow of water into a flow through a porous medium composed of ice. Herein, this mixture of ice and flowing water is referred to as naled slush.

During the experiments, naled slush eventually froze at its surface to form a solid crust of ice. Water continued to percolate through the layer of permeable slush, between the thickening ice crust over the slush and the underlying ice surface, until the permeability of the layer was sufficiently reduced to force water to flow over the frozen crust. Another slush layer then began to form. A second crust of ice would eventually form on the new surface flow. The continuing, cyclic process by which ice layers formed on the surface of slush layers resulted in the ice laminations that are a feature of naled ice. Depending on the amount of time that the surface layers of ice had for
thickening before they were covered by a fresh layer of naled slush, the laminations sometimes consisted entirely of solid layers of ice and sometimes consisted of alternating solid layers and layers of slush ice.

**Characteristics of Naled Slush**

Naled slush consists of a matrix of interlocked ice platelets. The growth of a slushy layer of naled ice is shown in Figure 6.1. This sequence of photographs illustrates the development of naled slush with time, from initially a few small platelets, to a denser matrix of larger platelets, and finally to a slush with a crusty surface of ice. The dark lines visible under the slush are metal seams, spaced about 4.4 cm apart, on the floor of the flume.

Water flowing over a naled becomes slightly supercooled before ice crystals form in the flow. In turbulent river flows, this supercooling is only a few hundredths of a centigrade degree; the supercooling that occurs in shallow, laminar flows over naleds is likely of the same order of magnitude, or less. An attempt, during the present study, to measure accurately the temperature of supercooling was not entirely successful because of the inherent difficulty of measuring a slight temperature difference that persists for a very short time in an extremely shallow flow. However, it is reasonable to assume that the supercooling was less than 0.1°C.

The literature on naleds appears to contain no observations concerning the form that ice crystals take when they grow in a slightly supercooled laminar flow over an ice base, but various investigators have discussed ice-crystal growth in slightly supercooled still water.
(a) a few relatively small ice platelets

(b) an increasing number of larger ice platelet

(c) finally, a frozen crust over the slush

Figure 6.1. Temporal development of naled ice slush.
Williamson and Chalmers (1966), for example, observed the growth of thin, disk-shaped crystals on a block of ice in slightly supercooled still water. The disks were attached to the block of ice at a point and retained their circular form as they grew.

As already mentioned, the ice crystals that formed on the ice base of a naled were observed to be platelets. They appeared to be attached at a point and may initially have been disk-shaped, but by the time they were large enough for their shape to be discerned, their upper boundary had been flattened by the free surface of the water. Beginning as small particles attached to the water-ice interface the ice platelets grew rapidly into the flow, reaching the water-air surface in a manner of minutes. Initially the platelets were distributed in complex, seemingly random, spatial arrays. The basal planes of the platelets were predominately vertical, or parallel to the general direction of the heat flux $\phi_{wa}$.

In many respects, the development of the slush on a naled's surface is similar to the growth, described by Weeks and Ackley (1982), of the transition zone that is associated with the beginning of columnar ice-crystal growth for sea ice. For example, the orientation of the platelets in the naled slush seemed to be governed by the process that Weeks and Ackley termed "geometric selection." In this process, crystals that grow vertically downward from an initial cover of sea ice enjoy a preferred orientation that enables them to grow rapidly ahead of crystals with orientations less favorable for growth. The crystals that are less favorably oriented are, essentially, crowded out. Because the free
surface of the water flow limited the vertical growth of the platelets, the slush development on the naleds did not progress to the columnar growth phase.

As the volumetric concentration of ice in the naled slush increased with time, the water flow over the naled was altered from a shallow free-surface flow to flow through a matrix, or porous medium, composed of ice crystals. The increasing impedance presented to the flow by the thickening ice matrix caused the upstream depth of water to significantly increase such that the naled surface remained wet even as the ice platelets continued to grow taller. It was evident from observations of dye injected into the flow that water passed over the platelets that constituted the ice matrix, near the water surface, as well as percolating among the platelets.

A Composite Description of Naled Ice Growth

What follows is a composite description of naled ice growth based on observations of laterally confined naleds grown in the laboratory flume; the description draws upon the foregoing discussions of the energy budget for naled ice growth and the characteristics of the naled slush.

The formation of a two-dimensional naled is portrayed sequentially in Figure 6.2, in which the vertical scale is distorted by approximately one order of magnitude. The cross sections in Figure 6.2 were deduced from elevation measurements that were taken at regular positions along the naleds grown in the refrigerated flume. Figure 6.2 represents a synthesis of the available data.
Figure 6.2. Growth of a two-dimensional naled.
Figure 6.2 --- continued.
In order to provide a perspective on the relative magnitudes of spread length and time associated with the different phases of naled ice growth, values of \( l/l_s \) and \( t/t_s \) that approximately delineate the different phases shown are included in Figure 6.2. The length scale \( l_s \) is defined by (4.32) and discussed in Chapter IV. The time scale \( t_s \) is defined as

\[
t_s = \frac{l_s^2}{100q_o}.
\] (6.1)

As discussed in Chapter IV, \( l_s \) is a more useful length scale than \( l_e \) for describing the layer-by-layer growth of a naled because the development of naled slush and the freezing of its surface is governed principally by \( \phi_{wa} \).

Naled ice growth was initiated in the flume by releasing water at the flume's upstream end. Figure 6.2a illustrates the initial conditions: an ice base covered the flume floor and the air and base temperatures were less than \( T_f \). Although the ice base in Figure 6.2a is horizontal, the ensuing description is not limited to ice bases that are initially horizontal. Nalees that developed on small slopes (\( S_o \leq 0.01) \) were observed to form in a similar manner.

The naleds developed initially in the manner illustrated in Figures 6.2b and 6.2c. The water flowed slowly downstream as a thin, sheet flow, cooled by heat transfer to the air, \( \phi_{wa} \), and to the ice base, \( \phi_i \), until the water temperature was reduced to \( T_f \). Inevitably, the water that had been exposed longest to the frigid air and ice base cooled to
Tf first. Consequently, ice platelets began appearing on the ice base near the downstream front. This occurred very soon after a naled was started (within several minutes in the experiments). As depicted in Figure 6.2c, within a short period of time (several minutes) after the first platelets appeared, ice platelets formed on the ice base throughout the flow except for a small region upstream, below the water source. Initially the platelets were distributed relatively sparsely and were small (see Figure 6.1a).

The next stage of naled development, depicted in Figure 6.2d, persisted for a significant period of time. The ice platelets in the naled slush continued to accumulate and grow during this period. The water flow was increasingly impeded so that the upstream water depth was increased and the rate of spreading, already slow, was decreased. The volumetric concentration of ice in the downstream slush was less than in the upstream slush because the ice base downstream had been covered by the flow for a shorter period of time than the ice base upstream.

As depicted in Figure 6.2e, eventually the surface of the naled slush began to solidify upstream. This marked the beginning of a continuous transition of the layer of naled slush into a solid layer of ice. The ice crust over the layer of naled slush became progressively thicker as the interstitial water in the slush froze. Meanwhile, the surface of the slush gradually solidified downstream. Eventually, the layer of slush could no longer pass the entire discharge and, therefore, some of the water flowed slowly onto the frozen crust, as depicted in Figure 6.2f. In effect, a fresh naled fed by a smaller discharge was now developing on the ice cover. The intermediate solid layer continued
to extend downstream until it reached the flume's floor, and froze to it, closing the flow path through the underlying slush. Upstream, where a fresh layer of slush covered the frozen crust, this intermediate solid layer could not thicken further because it was sandwiched between isothermal slush layers at temperature $T_f$. Downstream from the surface layer of slush, the ice crust continued to thicken. Because the slush was insulated from the frigid air by the ice crust, ice platelets could no longer accumulate and grow in the underlying slush layer.

A solid crust of ice eventually formed upstream over the uppermost layer of slush, as illustrated in Figure 6.2g. This solid crust then gradually extended downstream and thickened. The processes shown in Figures 6.2e, 6.2f, and 6.2g were repeated. A third slush layer developed on the surface of the second solid layer which, in turn, eventually froze to the ice base downstream. Then, yet another solid layer formed upstream (Figure 6.2h). Presumably, this process could continue indefinitely so that a large number of layers would develop.

The flume constrained each naled so that its initial growth was two-dimensional, as is attested by the uniformity of the naled surface across the width of the flume in Figure 6.3a. However, the surface of the slush always began freezing and solidifying in the central third of the flume's width rather than uniformly across the flume (Figure 6.3b). As the central portion hardened and thickened, constricting the flow through the underlying slush, the water was diverted to the sides of the flume so that the slush near the flume side-walls continued to develop. The subsequent naled surface was uneven, with a rough, dry center and
Figure 6.3. Views of a naled growing in the refrigerated flume.
wet, slushy sides. Strictly speaking, the naled was no longer two-di-

dimensional. The observed diversion of flow away from the regions of a
naled's surface that are solidifying is possibly a characteristic of the
growth of a naled in nature. In nature, as a naled's surface would
freeze, the source water would nearly always have the opportunity to
move laterally, away from the regions of frozen slush.

During the phase of naled ice growth depicted in Figure 6.2d, the
slush farthest upstream would sometimes become disconnected from the ice
base. When the base temperature, $T_b$, was just below the freezing tem-
perature, $T_f$, the buoyancy force of the ice platelets sometimes overcame
the attachment forces that anchored them to the ice base, and the entire
matrix over a short length of naled would ascend to, and float at, the
slowly rising water surface. New platelets would form on the ice base
under the floating slush canopy. This suggests that the forces anchor-
ing the platelets to the ice base possibly decrease with increasing $T_b$
(decreasing $\phi_1$).

**Ice Laminations**

Naled ice is usually laminated. Most often, the laminations found
in naturally occurring naleds have consisted of solid layers of ice.
However, unfrozen, slushy layers near the surface of a naled have been
reported, and thin, porous layers of ice crystals in zones that were
formerly occupied by liquid water, that has since been drained, are
sometimes found (Carey 1973). Solid layers of ice and layers of unfro-
zen slush were observed during the present study.
Figure 6.4 illustrates solid layers of ice in a piece of naled ice taken from a naled that was grown in the flume. Adjacent solid layers of ice developed in a naled at those positions where each successive slush layer on the naled's surface was exposed to the frigid air for a long enough time to permit the slush layer to freeze solid over its entire depth. A slush layer ceased solidifying when a new slush layer, spreading slowly downstream, covered the frozen surface of the underlying slush layer (thereby insulating it from $\phi_{wa}$). Consequently, adjacent solid layers of ice were likely to be found somewhat downstream from a naled's water source (see Figure 6.2h). Upstream, near a naled's water source, several alternating layers of solid and slush ice were sometimes found; a sample is shown in Figure 6.5. Because the surface slush layers spread very slowly downstream, the upstream region over which alternating layers of solid and slush ice were sometimes found was quite small relative to the spread length of a naled (the proportion of the naled in Figure 6.2h that has alternating layers is exaggerated for the sake of illustration).

Much of the time during which adjacent solid layers of ice were developing in a flume naled, a layer of unfrozen slush was present either on or directly under the naled's surface. As shown in Figure 6.6a, each solid layer of ice began as a slush layer on the surface of a naled. The slush layer gradually froze solid from the surface of the slush downward. An intermediate stage in the growth of a solid layer of ice is depicted in Figure 6.6b. The slush appears dark in this photograph because the ice was illuminated from behind.
Figure 6.4. Adjacent solid layers of ice in a piece of naled ice.

Figure 6.5. Alternating layers of solid and slush ice.
(a) laminated naled ice with a surface layer of slush

(b) naled ice with a partially frozen surface layer of slush

Figure 6.6. Slush layers on and just under the surface of naled ice.
CHAPTER VII
THE SPREADING OF LATERALLY CONFINED NALEDs

The spreading of two-dimensional, or laterally confined, naleds is influenced by six of the seven parameters in (4.51). That is,

\[ \frac{\ell}{\ell_e} = f \left( \frac{t}{t_e}, S_o, \phi_{wa}^*, \frac{\dot{\theta}_r}{\ell_e}, \frac{\theta}{\ell_e}, R_e \right). \] (7.1)

The influences of these six parameters are now discussed using the data obtained from the experiments.

The data show that during the early phase of a naled's growth, before the slush on its surface begins to freeze solid, \( \ell/\ell_e \) depends only on \( t/t_e, S_o, \) and \( \phi_{wa}^* \). After a transition time has passed, defined in terms of the time scale \( t_s \), the further spreading of a naled is influenced by \( \dot{\theta}_r \) and \( \theta/\ell_e \), in addition to \( t/t_e, S_o, \) and \( \phi_{wa}^* \). Although the growth of naled ice may also be influenced by variations in \( R_e \), the data show no discernable effect for small variations in \( R_e \).

As is evident in Table 5.1, most of the data were collected from naleds grown on initially horizontal \( (S_o = 0) \) ice bases. These data are presented and discussed in the following sections. The data for which the slope of the ice base was relatively steep, \( S_o = 0.01 \), are presented separately in the final section of this chapter.
Presentation of Data in Terms of Controlled Variables

The dimensionless parameters in (7.1) become more meaningful if the data are first examined in terms of the variables that were controlled during each experiment. These variables are included in

\[ l = f(t, q_0, \phi_{wa}, \phi_{io}, S_o). \]  \hspace{1cm} (7.2)

For the reasons discussed in Chapter V, the initial heat flux \( \phi_{io} \) is used rather than the diminishing heat flux, \( \phi_i \).

Figures 7.1, 7.2, and 7.3 illustrate spread length, \( l \), as a function of time, \( t \), for the flume naleds with \( S_o = 0 \). The data are separated into three groups in accordance with air temperature, \( T_a \).

Therefore, in each of Figures 7.1, 7.2, and 7.3, \( S_o \) is constant and \( \phi_{wa} \) is approximately constant. Consequently, the data in each figure should vary only with \( q_0 \) and \( \phi_{io} \).

It is shown in Figures 7.1 through 7.3, that the rate of naled spreading increases with increasing \( q_0 \) and decreases with increasing \( \phi_{wa} \) and \( \phi_{io} \). The heat fluxes \( \phi_{wa} \) and \( \phi_{io} \) slow a naled's spreading by causing ice to grow, which, effectively, removes water mass from the flow.

In addition, ice that grows to balance \( \phi_{wa} \) is in the form of ice platelets which impede the flow, thereby reducing the rate of spreading.

Comparison of, for instance, tests 1 (Figure 7.1) and 7 (Figure 7.3), in which \( q_0 \) and \( \phi_{io} \) have nearly the same values, verifies that the rates of spreading of the naleds decreased with increasing \( \phi_{wa} \). Comparison of tests 1 though 3, in which \( q_0 = 0.004 \) liters/second/meter, with tests 10
Figure 7.1. Nailed spread length related to time, average $T_a = -4.7^\circ C$. 

$T_a \sim -4.7^\circ C$, $\Phi_{wa} \sim 37 \ W/m^2$.  
$S_o = 0$ 

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Figure 7.2. Naled spread length related to time, average $T_a = -9.8^\circ C$. 

$T_o \sim -9.8^\circ C$, $\Phi_{wo} \sim 92$ W/m$^2$, $S_o = 0$
Figure 7.3. Naled spread length related to time, average $T_a = -13.2^\circ C$. 

$T_o \sim -13.2^\circ C, \Phi_w \sim 139 \text{ W/m}^2$. 

$S_o = 0$
through 12, in which \( q_0 = 0.008 \) liters/second/meter, shows that the
rates of spreading increased with increasing \( q_0 \). Also, a comparison of
tests for which \( q_0 \) is the same but \( \phi_{10} \) is different, such as tests 1
through 3 in Figure 7.1, reveals that the rates of spreading decreased
with increasing \( \phi_{10} \).

Figures 7.1, 7.2, and 7.3 illustrate that, qualitatively, all the
naleds spread in a similar manner. During their initial phase of growth
(as portrayed in Figure 6.2b), the naleds spread relatively quickly,
although their rates of spreading gradually decreased when ice platelets
grew and accumulated. As is indicated for tests 1, 2, and 20 in Figure
7.1, the naleds sometimes spread intermittently, stopping for periods of
time and then continuing. The spreading of a naled sometimes stopped
for relatively long periods of time when either its length approached
its time-varying equilibrium length (as was the case for test 1 in
Figure 7.1), or as the slush froze solid near the downstream end (as
occurred for tests 4 and 6 in Figure 7.2). Often, a fresh slush layer
would later spread over and beyond the downstream frozen surface,
thereby continuing the naled's expansion (this happened during tests 7
and 26 in Figure 7.3).

The data presented in Figures 7.1, 7.2, and 7.3 show some irregu-
larities because \( \phi_{wa} \) and \( \phi_i \) varied during each test (because the refrig-
eration units cycled on and off). These irregularities are manifested
as minor inconsistencies in the way in which data from various tests
separate from each other. For example, the curves through data from
tests that are different but have approximately the same \( q_0 \) and \( \phi_{wa} \).
should be separated cleanly in accordance with variations in $\phi_{io}$, but in
Figure 7.1, the curves for tests 1 and 2 intersect each other.
Similarly, in Figure 7.2 the curves for tests 4 and 6 intersect.
However, the minor inconsistencies in the data do not affect the prin-
cipal conclusions drawn from the study.
Besides the variations of $\phi_{i}$ due to the refrigeration cycles, the
ggradual decrease of $\phi_{i}$ due to naled ice growth was a matter of concern.
This concern was the motivation for tests 32 and 33, which were nearly
identical to tests 8 and 2, respectively, except that thicker initial
ice bases were used. The thicker the initial ice base, the smaller was
the effect of naled ice growth on the value of $\phi_{i}$. Therefore, $\phi_{i}$ dimin-
ished at a faster rate during tests 8 and 2 than it did during tests 32
and 33. As is illustrated in Figures 7.1 and 7.3, the spreading curve
for test 33 is similar to the spreading curve for test 2, and the
spreading curve for test 32 is similar to the spreading curve for test
8. This suggests that the data were not significantly influenced by the
ggradual decrease of $\phi_{i}$ that accompanied naled ice growth.

Presentation of Data in Terms of
Normalized Variables

Figures 7.4, 7.5, and 7.6 show the experimental data presented in
terms of $l/l_{eo}$ and $t/t_{eo}$. For all the data represented in these fig-
ures, $S_{o} = 0$. As in Figures 7.1, 7.2, and 7.3, the data are separated
into three groups in accordance with $T_{a}$, or, equivalently, $\phi_{wa}^{*}$. For
constant values of $S_{o}$ and $\phi_{wa}^{*}$, (7.1) implies that in each of Figures
7.4, 7.5, and 7.6, any separation of the data on naled spreading should
be consistent with variations in $\phi_{r}$, $w/l_{eo}$, and $R_{e}$. 
Figure 7.5. Normalized data, average $T_a = -9.8^\circ C$. 
Figure 7.6. Normalized data, $T_a = -13.2^\circ C$. 

$T_a \sim -13.2^\circ C$, $\Phi_{wa} \sim 7.2$, $S_\theta = 0$.
During the early phase of naled ice growth (portrayed in Figures 6.2b through 6.2d), before the surface of the slush layer freezes solid, the normalized rate at which a naled spreads is not affected by variations in $\Phi_r$, $w/l_{eo}$, or $R_e$ -- at least not for the ranges over which these parameters were varied during the experiments. This result is clearly illustrated in Figure 7.4 for the data that were collected from naleds grown in air temperatures of approximately $-4.7^\circ C$. During the experiments, the slush on these naleds did not freeze solid. Figure 7.4 reveals that these data, which separate into several curves in Figure 7.1, collapse approximately to form a single curve. Variations of $\Phi_r$, $w/l_{eo}$, or $R_e$ do not appear to affect this outcome.

The same conclusion can be drawn from Figures 7.5 and 7.6, in which the data from naleds grown under air temperatures of approximately $-9.8^\circ C$ and $-13.2^\circ C$ are plotted, respectively. The early phase of growth for these naleds was of relatively short duration. Consequently, in Figures 7.5 and 7.6, only the data for $t/t_{eo}$ less than about 0.05 form a single curve. To amplify the regions over which the normalized data fall together, $l/l_{eo}$ is plotted against $\log(t/t_{eo})$ in Figures 7.7, 7.8, and 7.9.

In Figures 7.5 and 7.6 (also 7.8 and 7.9) it can be seen that the data eventually diverge from the common curves that correspond to the early phase of naled ice growth. Although other influences may be present, the point of divergence for each data curve varies with $\Phi_r$. The larger the value of $\Phi_r$, the sooner, in terms of $t/t_{eo}$, the data diverge from the common curve below. However, if time is normalized by $t_s$, the
Figure 7.7. Semi-log plot of normalized data, average $T_a = -4.7^\circ C$. 

Symbols are defined in Figure 7.4.

\[ T_o \sim -4.7^\circ C, \; \phi_{wa}^* \sim 1.9, \; S_o = 0 \]
Figure 7.8. Semi-log plot of normalized data, average $T_a = -9.8^\circ C$. Symbols are defined in Figure 7.5.
Figure 7.9. Semi-log plot of normalized data, average $T_a = -13.2^\circ C$. 

Symbols are defined in Figure 7.6.
data suggest (as is discussed later in this chapter) that the divergence point in each case corresponds to a unique, transition value of approximately 0.035 for $t/t_s$. Apparently, near the transition value of $t/t_s$, the magnitude of the heat flux $\Phi_{wa}$ is effectively reduced over part of a naled's surface. Consequently, less of the water supplied to a naled freezes and, as is indicated in Figures 7.8 and 7.9, a naled's rate of spreading is increased relative to the common curve that represents the early phase of naled ice growth.

The exact reason for the effective reduction in $\Phi_{wa}$ when $t/t_s$ passes 0.035 is not clear from the data. However, because the transition time is in terms of $t_s$, this reduction must be related to the increasing accumulation of ice platelets. It seems that, as time increases beyond the transition value of $t/t_s$, an increasingly significant portion of the heat flux from a naled's surface acts to reduce the temperature within the existing ice platelets rather than acting to cause more ice to grow within the slush. This effect leads to a decrease in the rate at which ice platelets grow. The transition value of $t/t_s$ possibly represents the beginning of a continuous transition from the early, maximum rate of growth of ice platelets to the eventual slower rate of growth that occurs after the slush surface freezes over and partially insulates the underlying wet slush from the cold air above. If this is the case, the transition value of $t/t_s$ may correspond to the start of the process by which the slush freezes solid. In Chapter 8, some of these points are discussed further.
At and beyond their divergence points, the curves in Figures 7.5 and 7.6 (or 7.8 and 7.9) do not, in all cases, consistently separate in accordance with variations in $\phi_r$. This can be ascribed to three possible causes: the effect of Reynolds's number, $R_e$; experimental error; and the influence of lateral boundaries (flume side-walls).

The effect of $R_e$ would cause variations in the resistance to flow through the slush on a naled, and would be most significant during the early phase of naled ice growth. Once the slush on a naled's surface starts to freeze solid, the flow regime becomes significantly altered. However, the data indicate that the early phase of naled ice growth is not affected by variations in $R_e$. Nonetheless, it is possible that variations in $R_e$ that are much larger than the variations that occurred during the present experiments may influence naled ice growth.

Experimental data are always subject to experimental error. As is discussed in Chapter V, the heat fluxes $\phi_w$ and $\phi_i$ varied somewhat during the experiments. In addition, because of the uncertainties in the values of $T_b$ and $h_b$, it is possible that for some of the experiments the estimated values of $\phi_{io}$ are not very accurate. The influence of inaccuracy in the value of $\phi_{io}$ would be largest on data for which $\phi_{io}$ was a relatively large proportion of the total heat flux, $\phi_w + \phi_{io}$.

Therefore, the data in Figure 7.5 should be more affected than the data in Figure 7.6. Indeed, if one is expecting the data to separate in accordance with variations only in $\phi_r$, then the data in Figure 7.5 do appear to be less consistent than do the data in Figure 7.6.

Undoubtedly, the data are mildly affected by experimental error.
However, the scatter of data shows consistent variations when viewed in terms of $\hat{r}$.

The more likely explanation for the separation of the curves in Figures 7.5 and 7.6 is that, once the slush on the surface of a naled begins to freeze solid, lateral boundaries -- the flume's sidewalls -- affect the rate at which the naled spreads. The parameter $w/l_{eo}$ accounts, at least partially, for the influence of lateral boundaries. Examination of Figures 7.5 and 7.6 reveals that when two sets of data with nearly the same value of $\hat{r}$ are compared (such as tests 5 and 23 in Figure 7.5 or tests 17 and 26 in Figure 7.6), the data associated with lesser values of $w/l_{eo}$ lie above the data for which $w/l_{eo}$ is larger. In other words, the naleds for which the flume was relatively narrow spread faster than did the naleds for which the flume was relatively wide.

As described in Chapter VI, the slush on a naled usually froze first along the central one third of the flume's width. When this occurred, some of the upstream water flow spread slowly over the frozen region, while the remainder was diverted toward the flume's side-walls, away from the frozen region. The water in the slush near the flume's side-walls could flow past the frozen slush in the center of the flume and contribute to the spreading of the naled downstream. The observed increase in rate of spreading with decreasing $w/l_{eo}$ implies that the proportion of the total water discharge that was diverted towards the flume's side-walls increased with increasing source-water discharge (equivalent to decreasing $w/l_{eo}$).
Additionally, the lateral boundaries likely affected the distribution of $\phi_{wa}$ across the flume's width. It is noted in Appendix B that $\phi_{wa}$ is slightly smaller near the flume's side-walls than it is in the center of the flume. Besides being a function of the distance, $\ell$, between the side-walls, the distribution of $\phi_{wa}$ is also affected by the height of the side-walls.

It was common (see Figures 7.4 through 7.6) for a flume naled to spread beyond its initial equilibrium length, $\ell_{eo}$. The actual equilibrium length of a naled, $\ell_{e}$, increases with time, because $\phi_{i}$ decreases as a naled thickens. Additionally, a naled may spread beyond its equilibrium length when an ice crust freezes over the slush on the naled's surface and, thereby, insulates the slush from $\phi_{wa}$.

Figure 7.10 illustrates the influence of variations in $\phi_{wa}^*$ on the rate at which laterally confined naleds spread. During the early phase of their growths, the naleds that grew under colder air (larger $\phi_{wa}^*$) spread slower.

**Discussion of the Differences in the Normalized Rates of Spreading**

As shown in Figures 7.4 through 7.10, naleds spread at different rates for different values of $\dot{\phi}_{r}$ and $\phi_{wa}^*$. The effects on naled ice growth of both $\dot{\phi}_{r}$ and $\phi_{wa}^*$ can be attributed to two influences of $\phi_{wa}$ that are not accounted for by using $\ell_{eo}$ and $t_{eo}$ to normalize $\ell$ and $t$. First, the ice platelets that grow to balance $\phi_{wa}$ increase the resistance to water flow. This resistance is a function of $\phi_{wa}^*$, which determines the rate at which ice platelets grow, and of the Reynolds
Figure 7.10. Influence of $\phi_{wa}$ on the rate of fuel spreading.
number, \( R_e \). Second, \( \dot{\phi}_{wa} \) influences the time at which the slush on a naled begins to freeze solid. In terms of \( t/t_{eo} \), this time of freezing is not the same for different naleds because only \( \dot{\phi}_{wa} \), of the total heat flux, causes the slush to freeze. This influence of \( \dot{\phi}_{wa} \) is appropriately represented by \( \dot{\phi}_r \), which is the ratio of \( \dot{\phi}_{wa} \) to the total heat flux, \( \dot{\phi}_{wa} + \dot{\phi}_i \).

The influence on naled spreading of \( \dot{\phi}_r \) can also be explained by considering the division of a naled's total mass into unfrozen water, ice platelets, and ice accreting on the water-ice interface. At any given value of normalized time, \( t/t_{eo} \), two naleds that are supplied with different discharges of source water, \( q_o \), will consist of the same value of normalized total mass, \( M^* \). Normalized total mass per unit width of a naled is defined as

\[
M^*(t) = \frac{p_w q_o t}{p_w l_{eo}} = \frac{t}{100 t_{eo}}. \tag{7.3}
\]

For naleds that spread at the same normalized rate (\( l/l_{eo} \) as a function of \( t/t_{eo} \)), the scales \( l_{eo} \) and \( t_{eo} \) account for differences between various naleds in the division of the total mass into ice mass and water mass. For constant total heat flux, \( \dot{\phi}_{wa} + \dot{\phi}_i \), the proportion of \( M^* \) that consists of normalized ice mass, \( M_i^* \), is

\[
M_i^*(t) = \frac{M_i}{2} = \frac{\dot{\phi}_{wa} + \dot{\phi}_o}{p_w l} \left( t_{eo} \right) \int_{l_{eo}}^{l} \frac{t_{eo}}{t_{eo}} \left( t_{eo} - \frac{t_{eo}}{t_{eo}} \right) d\left( \frac{x}{l_{eo}} \right). \tag{7.4}
\]
in which $M_i = \text{ice mass per unit width of naled}$ and $t_e(x) = \text{time at which the spreading, downstream edge of a naled reaches a streamwise position x}$. The quantity $t_{eo}/k_{eo}$ is equal to $\rho w L/100(\phi_{wa} + \phi_i)$, so (7.4) can be reduced to an expression that, for given $t/t_{eo}$, depends only on the normalized rate of spreading (which is reflected in $k_{eo}$ and $t_{eo}$):

$$M_i^*(t) = \frac{1}{100} \int_{0}^{k_{eo}} \left[ \frac{t}{t_{eo}} - \frac{t_e(x)}{t_{eo}} \right] d(x). \quad (7.5)$$

Because the normalized mass of unfrozen water, $M_w^*$, is the difference between the total mass and the total ice mass, two naleds with the same values of $M^*$ and $M_i^*$ will also have the same $M_w^*$. While the total quantity of ice is accounted for, the scales $k_{eo}$ and $t_{eo}$ do not reflect the division of the ice into ice platelets and bottom ice. This division is reflected in the value of $r_e$.

Figures 7.4 through 7.10 show that, even though $r_e$ varies, the initial rates of spreading of different naleds are the same when $\phi_{wa}^*$ is the same. Naleds spreading at the same rate have the same values of $M_w^*$ and $M_i^*$, but different proportions of ice platelets and bottom ice. This implies that the resistance force $R_x$, which is a function of $\phi_{wa}^*$, is dependent on the growth rate of ice platelets, but not on the volumetric concentration of ice platelets in the slush.

The roles of $r_e$, $\phi_{wa}^*$, and $R_x$ are apparent in the non-dimensional depth-integrated flow equations (4.38, 4.42, 4.43, and 4.44). The ratio $r_e$ (which is equivalent to $k_{eo}/k_{s}$) is present in (4.43) and (4.44), in which it indicates the division of the normalized total mass of ice into
ice platelets and bottom ice. The parameters $\phi_{wa}^*$ and $R_e$ are included implicitly in (4.42), because they influence the resistance force, $\mathbf{R}_x$.

**Estimation of the Transition Time**

The transition time, which is equal to the time at which the data for a naled begins to diverge from the common curve for the early phase of naled ice growth, can be estimated from Figures 7.5 and 7.6, or from Figures 7.8 and 7.9. However, the transition time associated with each curve cannot be accurately determined because the data are sparse around each point of divergence.

A different approach to estimating the transition time for the data in each figure is to use plots of $q_o \frac{t}{l^2}$ against time, as shown in Figures 7.11 and 7.12. If the difference in density between water and ice is neglected, $q_o \frac{t}{l^2}$ is a geometric aspect ratio of a naled -- i.e., the average overall thickness of a naled, $q_o \frac{t}{l}$, divided by its length, $l$. The aspect ratio of a naled decreases sharply when the transition time is reached. Therefore, the time at which a naled's aspect ratio is a maximum can be used to obtain a more accurate estimate of the transition time. However, as can be seen in Figures 7.5 and 7.6, the sparsity of the data in Figure 7.11 still makes it difficult to estimate precisely the transition times. For example, for test 8 in Figure 7.11c, the transition time could be any time between $t/t_{eo} = 0.05$ and $t/t_{eo} = 0.22$. Figure 7.11, particularly 7.11c, reinforces the conclusion that the curves for naled spreading diverge in accordance with variations in $\phi_r$. 
Figure 7.11. Aspect ratios related to $t/t_{eo}$ (for a, b, and c, symbols are defined in Figures 7.4, 7.5, and 7.6, respectively).
Figure 7.12. Aspect ratios related to \( t/t_s \) (for a, b, and c, symbols are defined in Figures 7.4, 7.5, and 7.6, respectively).
In Figure 7.12, values of the aspect ratio $q_o t/t^2$ are plotted against time normalized by $t_s$. Rather than a consistent variation with $\dot{\Phi}_r$, the transition times are now scattered, apparently randomly, between $t/t_s = 0.02$ and $t/t_s = 0.05$. Therefore, a constant value, equal to 0.035 (which is the midpoint of the range), is assumed for the transition value of $t/t_s$. However, the scarcity of the data leaves open the possibility that the transition value of $t/t_s$ may vary slightly with $\dot{\Phi}_w$ or $\dot{\Phi}_r$, or both. In terms of $t/t_{eo}$, if the transition time normalized by $t_s$ is a constant equal to 0.035, then

$$\left(\frac{t}{t_{eo}}\right)_{critical} = \frac{0.035}{\dot{\Phi}_r^2}$$ \hspace{1cm} (7.6)

**Naleds Spreading over Mild Slopes**

Data from tests 28 and 30, for which $S_o = 0.01$ and $T_a = -4.7^\circ C$, are compared in Figure 7.13 with the curve that represents the average of the data from naleds grown for $S_o = 0$ and $T_a = -4.7^\circ C$. The data indicate that, all else equal, naleds grown on sloping surfaces spread faster than do naleds grown on horizontal surfaces.

For colder air, the data do not clearly indicate whether naleds spread faster on sloping than on horizontal surfaces. In Figure 7.14, the spread data for tests 29 and 31 are compared with average data from naleds grown on horizontal surfaces. During its early growth, the naled grown in test 31 spread significantly faster than did the naleds grown on horizontal surfaces. But, during its early growth, the naled grown
Figure 7.13. Normalized data, $S_o = 0.01$ and average $\phi_{wa}^* = 1.75$. 

- $\phi_{wa} \approx 1.9, S_o = 0$
- Test $\phi_{wa}$ w/ $t/t_{eo}$
  - 28 $\square R = 2.0, 0.31, 0.048, 2.3$
  - 30 $\triangle R = 1.5, 3.3, 0.022, 3.4$

- $S_o = 0.01$

$\frac{e}{\Theta}$ vs $\frac{t}{t_{eo}}$
Figure 7.14. Normalized data, $S_o = 0.01$ and average $\phi_{wa}^* = 6.7$. 
in test 29 spread only slightly faster than did the naleds grown on horizontal surfaces. During the later, layer-by-layer, phase of naled ice growth, the data in Figure 7.14 suggest that naleds grown on sloping surfaces may spread at about the same rates as naleds grown, under similar conditions, on horizontal surfaces. These data indicate that during the growth of an established naled (Figure 6.2h), the influence of $S_o$ is reduced.

The effect of small base slope on a naled's aspect ratio is illustrated in Figure 7.15. The aspect ratios of the naleds grown during test 29 and 31 decreased less sharply after the transition time than did the aspect ratios of naleds grown on horizontal surfaces. Figure 7.15 suggests that, for naleds grown on small slopes, the transition value of $t/t_s$ may be greater than 0.035. However, the data are insufficient to support any firm conclusions. Tests 28 and 30 were not sufficiently long for the transition time to be reached during these tests.
Figure 7.15. Aspect ratios of naleds grown on a small slope.
CHAPTER VIII
THE THICKENING OF LATERALLY CONFINED NALEDs

In this chapter, the experimental data on the form and thickening of two-dimensional, or laterally confined, naleds are discussed in terms of the key parameters given in (4.51). For any section of a naled, as shown in Figures 4.5 and 4.6, the thickness and depth measures of interest are total thickness, s; depth of slush layer, d; and thickness of ice laminations, \( \zeta \).

Additionally, in order to quantify the process whereby a layer of slush freezes and ice laminations form, the porosity, \( m \), and the water content, \( md \), of the surface slush layer are computed and discussed.

Each figure presented in this chapter contains data from naleds that were grown in the flume with a horizontal slope under approximately the same air temperature. The source-water discharge, \( q_o \), and the base heat flux, \( \phi_{io} \), varied from naled to naled. However, in terms of the normalized variables \( L/L_{eo} \) and \( t/t_{eo} \), the naleds represented in each figure all spread at the same rate.

**The Longitudinal Shape of a Naled**

The longitudinal shape of a naled is described by the overall thickness, \( s \), as a function of \( x \). The thickness \( s \) includes only the water and ice that accumulated following the start of a test; it does not include the thickness of the underlying initial ice base.
As is the case for the spreading of a naled, the thickening of a naled does not depend on \( \dot{r} \), \( w/l_{eo} \), or \( R_e \) for times less than the transition value of \( t/t_s \) (about 0.035). This is evident in Figures 8.1, 8.2, and 8.3. These figures illustrate the longitudinal shapes of the flume naleds for normalized times less than the transition time. The indicated values of \( t/t_{eo} \) are averages of the normalized times that are associated with the data on each curve. The point at which each curve intersects the x-axis corresponds to the appropriate spread length taken from either of Figures 7.7, 7.8, or 7.9.

It was pointed out in Chapter VII that, at any given value of \( t/t_{eo} \), the normalized total mass (ice and water) of one naled is the same as the normalized total mass of any other naled. Figures 8.1, 8.2, and 8.3 show that naleds spreading at the same normalized rate have the same longitudinal shape. That is, not only do they have the same normalized mass, but also the distribution of the mass over \( x/l_{eo} \) is the same.

Comparisons of Figures 8.1, 8.2, and 8.3 reveal that, for a given value of \( t/t_{eo} \), a naled's thickness relative to its length increases with increasing \( \phi_{wa} \). This is expected because, at a common value of \( t/t_{eo} \), different naleds have the same total mass but, as is established in Chapter VII, their normalized spread lengths decrease with increasing \( \phi_{wa} \). Equivalence in total mass corresponds approximately to equivalence in the cross-sectional areas under the curves that define the shapes of different naleds (neglecting differences, between naleds spreading at different rates, in the proportion of their total mass that
Figure 8.1. Longitudinal shape before transition time, average $T_a = -4.7^\circ{C}$. 

$T_a \sim -4.7^\circ{C}$, $\Phi_w^* \sim 1.9$, $S_o = 0$
Figure 8.3. Longitudinal shape before transition time, average $T_a = -13.2^\circ C$. 

$T_a \sim -13.2^\circ C, \Phi_{vo} \sim 7.2, S_0 = 0$

Test
- 7 O
- 18 △
- 17 ◊
- 18 ∗
- 25 ×
- 26 +
consists of ice, which is less dense than unfrozen water). Therefore, at the same value of $t/t_{eo}$, a shorter naled is thicker than a longer naled.

Figures 8.4 through 8.7 illustrate that, for times larger than the time associated with the transition value of $t/t_s$, the shape of a naled depends on $\dot{\phi}_r$ in addition to $t/t_{eo}$, $\dot{\phi}_{wa}$*, and $S_o$. It was shown in Chapter VII that, for larger values of $\dot{\phi}_r$, naleds spread faster and are both longer and thinner.

Because it influences the length of naled spreading, the parameter $w/k_{eo}$ also influences the longitudinal shape of a naled. However, the data in Figures 8.4 through 8.7 do not show this. Additionally, it is difficult to discern from the data the effect of $R_e$ variations on naled form.

As illustrated by the data associated with the largest values of $t/t_{eo}$ in Figures 8.4 through 8.7, the surface of a naled often becomes uneven after the slush on its surface begins to freeze solid.

**Thickness of Ice Laminations**

Each lamination of ice through the cross section of a naled results from a layer of slush that froze solid. At any location, comparisons of measured values of lamination thickness, $\zeta$, with measured values of slush depth, $d$, indicate that the thickness of an ice lamination is equal to the maximum depth attained by the slush from which the lamination developed. Therefore, the data on slush depth, which are discussed in the following section, can be used to predict the thicknesses of the ice laminations in a naled's cross section.
Figure 8.4. Longitudinal shape after transition time, $T_a = -9.8^\circ\text{C}$ and $\frac{t}{t_\infty} = 0.39$. 
Figure 8.5. Longitudinal shape after transition time, $T_a = -9.8\,^\circ C$ and $\phi_r = 0.88$. 

$T_a = -9.8\,^\circ C$, $\phi_{eo} \approx 4.8$, $S_o = 0$.
Figure 8.6. Longitudinal shape after transition time, $T_a = -13.2^\circ C$ and $\Phi_r = 0.50$. 
Figure 8.7. Longitudinal shape after transition time, $T_a = -13.2^\circ C$ and $\Phi_r = 0.92$. 

$T_a \sim -13.2^\circ C$, $\Phi_{wo} \sim 7.2$, $S_o = 0$. 

Test $g \Delta .89$ 
18 $\times .92$ 
27 $\bullet .94$ 

$X'_{eo} \sim 0.52$ 

$X'_{eo} \sim 0.083$ 

$0.083$ 

$0.21$ 
$0.22$ 
$0.3$ 
$0.8$ 
$1.2$ 
$1.5$
Depth and Porosity of Naled Slush

Naled ice grows as successive layers of slush that form and freeze solid. When a slush layer freezes, water spreads over its frozen, crusty surface and a new slush layer forms. Besides being influenced by each of the key parameters given in (4.51), the depth and porosity of a layer of naled slush may be affected by its position in the sequence of formation of slush layers. The data presented in this section were taken from the first layer of slush that developed after naled ice growth was initiated in the refrigerated flume.

In dimensionless terms, the depth of the initial layer of slush, \( d/L_{eo} \), equals the thickness of the naled, \( s/L_{eo} \), minus the thickness of accumulated bottom ice, \( \eta/L_{eo} \). The normalized depth of slush varies with \( x/L_{eo} \), \( t/t_{eo} \), and \( \phi_{wa} \) in essentially the same manner as does \( s/L_{eo} \). That is, slush depth decreases with increasing \( x \) and increases with increasing \( t \). The depth of slush on a naled increases with increasing \( \phi_{wa} \), conversely to the effect of \( \phi_{wa} \) on the length of naled spreading.

Until an ice crust forms over the initial layer of slush, the depth of the slush can be determined at any time and at any longitudinal position if \( s/L_{eo} \) is estimated from Figures 8.1 through 8.7. Using (4.33) with \( \phi_i \) assumed to be constant, \( d/L_{eo} \) may be expressed as

\[
\frac{d}{L_{eo}} = \frac{s}{L_{eo}} - \frac{\rho_w}{100 \rho_i} (1 - \phi) \left( \frac{t}{t_{eo}} - \frac{t_{eo}}{t_{eo}} \right). \tag{8.1}
\]
This equation shows that \( d/\ell_{eo} \) increases with increasing \( \dot{\varphi}_r \). Even during the early phase of naled ice growth, \( d/\ell_{eo} \) depends on \( \dot{\varphi}_r \) in addition to depending on \( x/\ell_{eo} \), \( t/t_{eo} \), \( \dot{\varphi}_{wa}^* \), and \( S_0 \).

The maximum depth of slush ice on a naled is equal to the thickness of the ice laminations in a naled's cross section. Table 8.1 indicates the maximum depths that were reached by slush layers near the upstream end of the flume. The values of \( d/\ell_s \) that are associated with \( \dot{\varphi}_{wa}^* = 4.8 \) and \( 7.2 \) were obtained directly from data. The values of \( d/\ell_s \) for \( \dot{\varphi}_{wa}^* = 1.9 \) were estimated using (8.1) and Figure 8.1. The time at which the slush upstream reached a maximum depth ranged from \( t/t_s = 0.06 \) to \( t/t_s = 0.10 \). At some time within this range, the surface of the slush started to freeze solid and, thereby, stopped the thickening of the initial layer of slush.

If, as in Table 8.1, \( d \) is normalized with \( \ell_s \) rather than with \( \ell_{eo} \), and \( t \) is normalized with \( t_s \) rather than with \( t_{eo} \), the maximum slush depth and the time at which it occurs vary little with \( \dot{\varphi}_r \). In fact, the quantity of data is not sufficient to identify any consistent variations with \( \dot{\varphi}_r \) in either the maximum value of \( d/\ell_s \) or in the corresponding value of \( t/t_s \). However, using (8.1) and Figures 8.1 through 8.7, it can be shown that the maximum value of \( d/\ell_s \) and the associated value of \( t/t_s \) cannot both be independent of \( \dot{\varphi}_r \) -- one or the other, or both, must be dependent on \( \dot{\varphi}_r \).

The values of porosity given in Table 8.1 were computed using the following non-dimensional form of (4.30):
Table 8.1

Maximum Depth of Slush Upstream* and Porosity, m (0.06 < t/t_s < 0.10).

<table>
<thead>
<tr>
<th>( \phi_{wa} )</th>
<th>100d/( l_s )</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>0.10</td>
<td>0.27</td>
</tr>
<tr>
<td>4.8</td>
<td>0.20</td>
<td>0.45</td>
</tr>
<tr>
<td>7.2</td>
<td>0.26</td>
<td>0.58</td>
</tr>
</tbody>
</table>

* \( x/l_{eo} < 0.1 \)

\[
m = 1 - \frac{\rho_w l_s}{100\rho_i (\frac{d}{d})} (\frac{t}{t_s} - \frac{t}{t_s}).
\]

(8.2)

The depths of slush included in Table 8.1 are for the upstream end of a naled, where \( t / l \) can be taken as zero. The ranges of \( m \) that are given in Table 8.1 suggest that the value of porosity that is associated with the maximum upstream slush depth increases with \( \phi_{wa}^{*} \).

The maximum depth of slush decreased with increasing distance over a naled, \( x \), but enough data to describe this variation is available only for \( \phi_{wa}^{*} = 7.2 \). The following measurements were made for \( \phi_{wa}^{*} = 7.2 \):

at \( x/l_{eo} \) near 0.5, the maximum slush depth (and lamination thickness) ranged from about \( d/l_s = 0.23 \) to \( d/l_s = 0.29 \) and at \( x/l_{eo} \) near 1.0, it ranged from about \( d/l_s = 0.14 \) to \( d/l_s = 0.23 \).

Table 8.2 lists ranges for the values of upstream depth and corresponding porosity of the slush that formed on the naleds when \( t/t_s \) was 0.035. Even though the transition that occurs when \( t/t_s \) passes 0.035 is
related to the accumulation of ice platelets within the slush, this
transition time coincides with neither a unique value of \( m \) nor of \( \frac{d}{\ell_s} \).

Table 8.2
Depth of Slush Upstream* and Corresponding
Porosity, \( m \), at \( \frac{t}{t_s} = 0.035 \).

<table>
<thead>
<tr>
<th>( \phi ) ( w_a )</th>
<th>( 100d/\ell_s )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>0.06 -- 0.13</td>
<td>0.36 -- 0.71</td>
</tr>
<tr>
<td>4.8</td>
<td>0.14 -- 0.17</td>
<td>0.73 -- 0.78</td>
</tr>
<tr>
<td>7.2</td>
<td>0.17 -- 0.20</td>
<td>0.78 -- 0.81</td>
</tr>
</tbody>
</table>

* \( x/\ell_{eo} < 0.1 \)

The slush layers that followed the initial layer were nearly as
deep as the initial layer. This observation suggests that little water
flowed through slush layers under a naled's surface. The succeeding
layers formed with nearly as much water flow as did the initial layer.

Quantity of Unfrozen Water in Naled Slush

The proportion of the slush on a naled that consists of unfrozen
water is represented by the porosity, \( m \). The product \( md \) is the quantity
of water in a volume of slush, \( d \) units deep, that underlies one square
unit of the surface of the slush.

For the same value of \( \frac{t}{t_{eo}} \), naleds spreading at the same normal-
ized rate contain the same normalized mass of unfrozen water. Figures
8.8, 8.9, and 8.10 illustrate that the distribution over \( x/\ell_{eo} \) of the
unfrozen water mass is also the same. The values of porosity needed to obtain the ordinates in these figures were computed using (8.2). All of the data in Figures 8.8 through 8.10 are associated with naled ice growth before the transition time has passed. As in Figures 8.1 through 8.7, the points at which the curves in Figures 8.8 through 8.10 intersect the x-axis coincide with the appropriate spread lengths taken from Figures 7.7 through 7.9.

The quantity of unfrozen water in the slush on a naled's surface stops increasing after the spread length approaches $\ell_{eo}$. This result, which is evident in Figure 8.8, verifies the assumption made in Chapter IV that, when a naled's spread length reaches its equilibrium length, $\dot{\ell}(md)/\dot{\ell}t = 0$. The reason that the flume naleds continued to spread farther after their lengths reached $\ell_{eo}$ is that their actual equilibrium lengths increased with time because $\phi_i$ decreased with time.
Figure 8.8. Unfrozen water in slush, average $T_a = -4.7^\circ C$. 

$T_o \sim -4.7^\circ C$, $\Phi_{wo}$ $\sim 1.9$, $S_o = 0$
Figure 8.9. Unfrozen water in slush, average $T_a = -9.8^\circ C$. 

$T_a \sim -9.8^\circ C, \Phi_{wo} \sim 4.8, S_o = 0$
Figure 8.10. Unfrozen water in slush, average $T_a = -13.2^\circ C$. 

$T_a \approx -13.2^\circ C$, $\phi_{wa} \approx 7.2$, $S_o = 0$

Test 7 O △ 16 ◆ 18 ◊ 25 X 26 +

$\frac{\theta}{\theta_{eo}}$, $\frac{X}{\ell_e}$ x

$100 \text{md}_e^{\circ}$

$\frac{100 \text{md}_e^{\circ}}{100 \text{md}_e^{\circ}}$
CHAPTER IX
APPLICATIONS OF RESULTS

Under various assumptions, the data in Chapters VII and VIII can be used to predict approximately the growth of a naled in nature. Ideally the naled being studied would be growing on a nearly horizontal surface, be laterally confined, be supplied at a single location by a steady discharge of source water that is distributed evenly over the naled's width, and be subject to constant heat fluxes $\Phi_w$ and $\Phi_i$.

In nature, although the conditions of steady discharge of source water and steady heat fluxes are probably never satisfied entirely, the geometric constraints may be satisfied approximately by some river naleds, and also by naleds that grow in culverts and other drainage facilities. The ice cover over which a river naled grows may be relatively smooth and level. Furthermore, at least until it thickens enough to overflow the river banks, a river naled may be laterally confined. However, the width between the lateral boundaries would probably vary along the spread length of a river naled, and the lateral boundaries are not likely to be vertical.

In order to apply the data in Chapters VII and VIII, for which the independent variables were approximately constants, it would be necessary to use averages of the parameters that, in nature, vary with time and location. For a laterally confined river naled, an average width
could be identified. A time and space-averaged value for $\phi_{wa}$, which varies with air temperature, wind speed, and cloud cover, would be required. In most cases, the heat flux $\phi_i$ could be assumed to be negligibly small; that is, it can be assumed that $\phi_i = 1.0$.

Perhaps the largest difficulty in applying the data to naled ice growth in nature lies in the specification of an appropriate value for the source-water discharge rate and of the location at which the source water is discharged. Naled ice growth in a river may be fed simultaneously by several discharges of source water at various locations. The total discharge of source water is unlikely to be steady or distributed evenly across the width of a naled. It may be difficult to estimate or measure the magnitude of the source-water discharge. It may also be necessary to adjust the source-water discharge to account for the water mass that is added to or removed from a naled because of precipitation, evaporation, and condensation.

An additional difficulty is that, for naleds that pose engineering problems, the values of the independent parameters in (4.51) are not likely to all be within the ranges over which these parameters were varied in the experiments. Because of wind and colder air, the heat flux $\phi_{wa}^*$ from a naled in the Sub-Arctic would be typically much larger, perhaps two or three times larger, than $\phi_{wa}^*$ from a naled in the refrigerated room. Also, the Reynolds number, $Re$, is likely to be much larger for a river naled than for a flume naled. Furthermore, river naleds probably grow for longer periods of time, in terms of $t/t_{eo}$, than did the flume naleds. This is suggested by the many laminations in the
rivers naled illustrated in Figure 3.8 compared with the maximum number of laminations that were observed in the laboratory, illustrated in Figure 6.4.

Clearly, before naled ice growth can be reliably predicted, much work remains to be done both in the laboratory and in the field.
CHAPTER X
CONCLUSIONS AND RECOMMENDATIONS

The major contributions of this study are the developments of a theory and a detailed description of naled ice growth. The study involved theoretical formulation and dimensional analysis of naled ice growth, as well as an extensive set of laboratory experiments using a refrigerated flume.

Although most of the observations and results from the study apply to naled ice growth in general, the study was primarily aimed at elucidating and formulating the growth of two-dimensional, or laterally confined, naleds.

The following principal conclusions are drawn from the study:

(1) Naled ice growth occurs as a cyclic process in which a layer of ice-water slush forms over a frozen base composed of successive layers of solidified ice slush. The most recent layer of slush freezes over and becomes, in turn, part of the frigid base over which a subsequent layer of slush forms. All the while, the naled spreads downstream and thickens. The cyclic nature of naled ice growth is illustrated in Figure 6.2.

(2) A physically meaningful length scale for describing naled ice growth is the spread length that is termed the equilibrium length, \( l_e \), and defined as
\[ \ell_e = \frac{q_o \rho_w L}{\phi_w a + \phi_i} \]  

(4.31)

in which \( q_o \) = discharge of source water per unit width of naled, \( \rho_w \) = mass density of water, \( L \) = latent heat of fusion for water, \( \phi_w a \) = heat flux from the surface of a naled to the air above, and \( \phi_i \) = heat flux from a naled into the underlying frigid base (the heat fluxes and source-water discharge are illustrated in Figure 4.1). For length of spread equal to \( \ell_e \), the discharge of source water to a naled is equal to the rate at which water is freezing over a naled.

(3) Associated with the length scale \( \ell_e \) is the time scale \( t_e \), which is defined as

\[ t_e = \frac{\ell_e^2}{100q_o}. \]  

(10.1)

The significance of \( t_e \) is that it is the time required to cover a unit width of naled surface, \( \ell_e \) in length, with water of average depth \( \ell_e/100 \).

(4) Figures 7.1, 7.2, and 7.3 illustrate that, qualitatively, all the naleds spread in a similar manner. During their initial phase of growth (as portrayed in Figure 6.2b), the naleds spread relatively quickly, although their rates of spreading gradually decreased as ice platelets grew and accumulated in the surface layer of slush. As is indicated for tests 1, 2, and 20 in Figure 7.1, the naleds sometimes spread intermittently, stopping for periods of time and then continuing. The spreading of a naled sometimes stopped for relatively long periods.
when either its length approached its time-varying equilibrium length (as was the case for test 1 in Figure 7.1), or as the slush froze solid near the downstream end (as occurred for tests 4 and 6 in Figure 7.2). Often, a fresh slush layer would later spread over and beyond the downstream frozen surface, thus continuing the naled's expansion (this happened during tests 7 and 26 in Figure 7.3).

(5) The many factors influencing naled ice growth can be reduced to a set of seven key independent, dimensionless parameters:

- \( x/l_e \) = normalized streamwise position along a naled;
- \( t/t_e \) = normalized time;
- \( S_o = \) slope of frigid base under a naled;
- \( \phi_w = \phi_{wa}/\rho_w L^{3/2} = \) normalized heat flux from a naled's surface;
- \( \phi_r = \phi_{wa}/(\phi_{wa} + \phi_1) = \) heat flux ratio;
- \( w/l_e = \) normalized flume width (represents boundary effects due to flume side-walls);
- \( R_e = q_o/\nu = \) Reynolds number (\( \nu = \) kinematic viscosity of water).

The variables describing the state of a naled, and dependent on the key independent parameters, are listed in Table 4.1 and illustrated in Figure 4.6. Included in these variables are a naled's spread length, \( l \), and overall thickness, \( s \).

(6) A transition time is identified which apparently coincides with the beginning of the processes by which the initial layer of ice-water slush on a naled freezes solid. In terms of the time scale \( t_s \) (defined in (6.1)), this transition time is approximately \( t/t_s = 0.035 \). However, this value of \( t/t_s \) may vary slightly with \( \phi_w \) and \( \phi_r \), and significantly with \( S_o \).
(7) The early phase of naled ice growth, before the transition time has passed, is influenced by only four of the seven key parameters. These four significant parameters are $x/t_e$, $t/t_e$, $S_o$, and $\phi_{wa}^*$. For the same values of $S_o$ and $\phi_{wa}^*$, naleds spread at the same normalized rate (shown in Figures 7.7 through 7.9). Also, as illustrated in Figures 8.1 through 8.3, all of the naleds had the same normalized shape. For large values of $\phi_{wa}^*$, the naleds spread more slowly and, consequently, were shorter and thicker than were the naleds growing under smaller values of $\phi_{wa}^*$. The small amount of data collected from naleds growing on sloping surfaces suggest that, during the early phase of growth, they spread faster on sloping surfaces than on horizontal surfaces (see Figures 7.13 and 7.14).

(8) After the transition time has passed, two of the remaining three parameters -- $\phi_r$ and $w/t_e$ -- come into play. As illustrated in Figures 7.5 and 7.6, the normalized rate of spreading of a naled increases with increasing $\phi_r$ and with decreasing $w/t_e$. Figures 8.4 through 8.7 show that naleds growing under large $\phi_r$ are both longer and thinner at a given value of $t/t_e$ than are naleds growing under smaller $\phi_r$. The data from naleds growing on sloping surfaces suggest that after the transition time has passed, the rate at which a naled spreads may no longer depend on the slope of the surface over which it is spreading (see Figure 7.14).

(9) Although it is possible that the Reynolds number, $R_e$, also influences naled ice growth, for the relatively narrow range of Reynolds numbers attained during the experiments, no consistent effect due to variations of $R_e$ can be discerned.
The thickness of the ice laminations in a naled correspond to the thickness attained by slush layers before they freeze solid.

Recommendations for Further Research

The results of the present study suggest many topics for future research on naled ice growth. Some of these topics are listed below. Suggestions are offered for further study of laterally confined, or flume, naleds and for study of naleds with different geometries.

Under improved experimental conditions, much could be learned by repeating some of the experiments of the present study. For instance, if $\phi_w$ and $\phi_i$ could be maintained more nearly constant and if their values could be determined more precisely, it would be possible to determine whether the transition time in terms of $t/t_s$ is truly a constant or whether it varies predictably with several of the parameters listed in (4.51). It would also be useful to devise a method for measuring the number and thicknesses of the laminations in naled ice, including the surface layer of slush, without needing to cut into the naled. Such a device would measure elevation differences more precisely than does a point gauge, and it would permit the freezing of the slush layers to be monitored. Perhaps a device that transmits acoustic signals and receives the reflections (such as an "acoustic flaw detector") would be suitable for this purpose. If computerized data acquisition techniques were employed, data could be collected at more frequent intervals during a naled's growth.
As noted above, for nalesds in nature the independent parameters listed in (4.51) are likely to have values that are outside of the ranges over which these parameters varied during the present experiments. This suggests the following studies on naled ice growth:

1. on various slopes (variations of $S_o$);
2. in colder air (larger $\phi_{wa}^*$);
3. in wider and narrower flumes (variations in $w/l_{eo}$);
4. with larger discharges of source water (larger $q_o$ and thus larger $R_e$);
5. for longer periods of growth (larger $t/t_{eo}$).

Because, in nature, $\phi$ generally is small compared with $\phi_{wa}$, future studies should focus on naled ice growth under relatively large values of $\phi_r$, say $0.5 < \phi_r < 1.0$.

Before a numerical model of laterally confined naled ice growth can be constructed the following information is needed:

1. The magnitude of the resistance term $\bar{R}_x$ as a function of the parameters in (4.51).
2. The boundary condition that is associated with the advancing, downstream front of a naled.

It would also be interesting to study the shear force that acts on water flowing over a smooth ice base. During the present study, a brief attempt to determine this force suggested that it may be significantly larger than the shear force acting on flow over a smooth surface on which no phase change is taking place.
Of course, naleds of many different geometries can be studied. Two relatively simple geometries can be studied using a point discharge of source water. Naleds that are radially symmetrical can be grown by discharging source water at the center of a flat surface, such that the water can spread in all directions away from its source. This would produce a circular naled. Although boundary effects would be introduced, one quadrant of a circular naled could be grown by discharging source water in the corner of a rectangular basin. A naled in the shape of a fan could be grown from source water discharged at a point on a sloping flat surface, such that the water would spread down the slope.

Much remains to be learned about naled ice growth. Besides the laboratory investigations discussed above, field studies are needed to provide a basis of comparison for the experimental data. Naled ice growth is a subject that may keep interested researchers busy for many years to come.
APPENDIX A

DEPTH-INTEGRATED EQUATIONS FOR FLOW OF WATER OVER THE SURFACE OF A NALED

The equations derived below are based on the principles of conservation of mass, momentum, and thermal energy applied to free-surface single-phase flow of water over an ice base. In order to calculate \( \eta(x,t) \) and \( T_w(x,t) \) (see Figure 4.1) in the upstream region of water flow over a two-dimensional naled, the depth-integrated equations can be solved simultaneously with (4.5), subject to (4.1) and (4.3) as boundary conditions.

The flow over the surface of a growing naled is laminar, but, for generality, the equations presented here have been derived such that they apply to both laminar and turbulent flows. For turbulent flows, the dependent variables \( u, v, p, \) and \( T \) -- the horizontal and vertical components of velocity, the pressure, and the water temperature -- represent time-averaged quantities.

**Conservation of Mass**

The conservation-of-mass equation for two-dimensional flow of water is

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{A.1}
\]
Integration of (A.1) over the depth of water flow, as illustrated in Figure A.1, yields

\[ \frac{\partial q}{\partial x} + \frac{\partial d}{\partial t} + \frac{p_i}{p_w} \frac{\partial n}{\partial t} = 0, \]  

(A.2)

in which the kinematic boundary conditions, (4.17) and (4.18), and relations (4.15) and (4.16) have been used. In words, (A.2) states that any change in water discharge with distance x is balanced by a change in the water depth with time (change in local water storage) plus a change in the elevation of the ice base with time (water taken from the flow to grow ice).

By assuming that \( m = 1 = \text{constant}, \) equations (A.1) and (A.2) can be obtained directly from (4.8) and (4.19), respectively.

Figure A.1. Definition sketch for integration over water depth.
Conservation of Momentum

The time-averaged form of the Navier-Stokes equations represent the conservation of momentum principle in an incompressible fluid. After adding (A.1) to each left-hand side of the two-dimensional Navier-Stokes equations, they can be written as

\[
\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} = - \frac{1}{\rho_w} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[(\nu+\varepsilon_x)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\nu+\varepsilon_y)\frac{\partial u}{\partial y}\right] \tag{A.3}
\]

and

\[
\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} = - \frac{1}{\rho_w} \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[(\nu+\varepsilon_x)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\nu+\varepsilon_y)\frac{\partial v}{\partial y}\right] - g. \tag{A.4}
\]

in which the eddy viscosity concept has been used to model the Reynolds stresses (\(\varepsilon_x \text{ and } \varepsilon_y \) = eddy viscosity coefficients).

Under the assumption of hydrostatic pressure (equation 5.22), only (A.3) need be integrated over the depth of water flow. Because, for relatively shallow free-surface flows, the velocity gradient \(\partial u/\partial y\) can be expected to be much larger than the velocity gradient \(\partial u/\partial x\), the second term on the right-hand side of (A.3) is negligibly small compared with the third term on the right-hand side. Using (4.15) through (4.18), together with (4.21) and (4.23), integration of (A.3) over the flow depth yields
\[
\frac{\partial q}{\partial t} + \frac{\partial (\beta q^2/d)}{\partial x} + gd\frac{\partial h}{\partial x} + \frac{\tau_o}{\rho_w} = 0, \tag{A.5}
\]
in which the additional boundary conditions

\[
(v + \epsilon_y) \frac{\partial u}{\partial y} = 0 \quad \text{at } y = h \tag{A.6}
\]
and

\[
(v + \epsilon_y) \frac{\partial u}{\partial y} = \frac{\tau_o}{\rho_w} \quad \text{at } y = y_b \tag{A.7}
\]
have been invoked. The bottom shear stress, \( \tau_o \), may be modeled using the Darcy-Weisbach resistance coefficient. However, this resistance coefficient may not be applicable because of the unknown effect of the ice growth, or melt, at the water-ice interface on the shear stress at the interface.

**Conservation of Thermal Energy**

After adding (A.1) to the left-hand side, the principle of conservation of thermal energy for two-dimensional flow may be written as

\[
\frac{\partial T}{\partial t} + \frac{\partial (uT)}{\partial x} + \frac{\partial (vT)}{\partial y} = \frac{\partial}{\partial x}[\left(\alpha + \alpha_{tx}\right) \frac{\partial T}{\partial x}] + \frac{\partial}{\partial y}[\left(\alpha + \alpha_{ty}\right) \frac{\partial T}{\partial y}] + \frac{\delta}{\rho_w C_w}, \tag{A.8}
\]
in which \( T \) = local water temperature, \( \alpha_w \) = thermal diffusivity of water, \( \alpha_{tx} \) and \( \alpha_{ty} \) = direction-dependent thermal turbulent diffusivity coeffi-
cients, $C_w$ = specific heat of water, and $\phi$ = viscous dissipation function.

Before (A.8) is integrated over the water depth, the following depth-averaged quantities need to be defined:

$$\overline{T} = \frac{1}{d} \int \frac{h}{y_b} T dy,$$  \hspace{1cm} (A.9)

$$\overline{uT}_w = \frac{1}{d} \int \frac{h}{y_b} uT dy = \xi \overline{T},$$  \hspace{1cm} (A.10)

and

$$\overline{\psi} = \frac{1}{d} \int \frac{h}{y_b} \psi dy,$$  \hspace{1cm} (A.11)

in which $T_w = \text{bulk water temperature}$ and $\xi = \text{a mean temperature correction factor}$. For a uniform, free-surface laminar flow, the factor $\xi$ is approximately 1.25. For turbulent flow $\xi$ should be approximately equal to 1.0. For simplification it is assumed that $\xi = 1.0$ or, equivalently, that $T_w = \overline{T}$.

In addition to the kinematic boundary conditions, (4.17) and (4.18), the following boundary conditions, which equate the resultant heat fluxes within the fluid at the boundaries of the flow to the heat fluxes from the boundaries, are used:
\[ \rho_w C_w [\left( \alpha_w + \alpha_{tx} \right) \frac{\partial T}{\partial x} \frac{\partial h}{\partial x} - (\alpha_w + \alpha_{ty}) \frac{\partial T}{\partial y}] = \phi_{w0} \text{ at } y = h \quad \text{(A.12)} \]

and

\[ \rho_w C_w [\left( \alpha_w + \alpha_{tx} \right) \frac{\partial T}{\partial x} \frac{\partial y}{\partial x} - (\alpha_w + \alpha_{ty}) \frac{\partial T}{\partial y}] = - \phi_{wi} \text{ at } y = y_b. \quad \text{(A.13)} \]

After application of (A.9) through (A.13) and the assumption that \( \xi = 1.0 \), the result of integration of (4.39) over the depth of flow is

\[ \frac{\partial (\phi T_w)}{\partial t} + \frac{\partial (\phi T_w)}{\partial x} \]

\[ = \frac{\partial}{\partial x} \left[ d(\alpha_w + \alpha_{tx}) \frac{\partial T}{\partial x} \right] - T_f \frac{\rho_i}{\rho_w} \frac{\partial h}{\partial t} - \frac{\phi_{w0} + \phi_{wi}}{\rho_w C_w} + \frac{d \phi}{\partial x}. \quad \text{(A.14)} \]

For relatively shallow, free-surface flows, it can be shown that

\[ \left( \alpha_w + \alpha_{tx} \right) \frac{\partial T}{\partial x} = \left( \alpha_w + \overline{\alpha}_{tx} \right) \frac{\partial T}{\partial x} = \frac{\partial T_{T_x}}{\partial x}, \quad \text{(A.15)} \]

in which \( F_x \), a longitudinal dispersion coefficient, replaces \( \left( \alpha_w + \overline{\alpha}_{tx} \right) \).

In laminar flow, \( F_x = \alpha_w \). It can also be shown that

\[ \overline{\psi} = \frac{\overline{uT_0}}{d}. \quad \text{(A.16)} \]

By subtracting (A.1), using (4.5) to eliminate \( \partial h/\partial t \) and (4.16) to replace \( \overline{u} \) with \( q \), and substituting (A.15) and (A.16), equation (A.14) can finally be written as
\[
\frac{\partial T_w}{\partial t} + \frac{q}{d} \frac{\partial T_w}{\partial x} + \phi_w \frac{\partial T_w}{\partial x} - \frac{(T_w - T_e)(\phi_i - \phi_w)}{\rho_w C_w} - \frac{\tau q}{\rho_w C_w d^2} = \frac{\partial}{\partial x} \left( d \phi F \frac{\partial T_w}{\partial x} \right).
\] (A.17)

Local and convective changes in $T_w$ are balanced by the net vertical heat flux from the flow (third term), heat added to the flow by friction (fifth term), and longitudinal dispersion of the temperature (right-hand side). The fourth term accounts for the increase in the mean temperature of the flow that results when low temperature water at the water-ice interface is removed from the flow to grow ice.
APPENDIX B

DETERMINATION OF THE WATER SURFACE HEAT-TRANSFER COEFFICIENT IN THE LOW TEMPERATURE FLOW FACILITY OF THE IIHR

Experiments were conducted in the refrigerated room of the Low Temperature Flow Facility to determine the coefficient, $h_{wa}$, of heat transfer from a water surface to overlying air (see (4.1) and Figure 4.1). Two sets of experiments were performed. The aim of the first set of experiments was to measure the variation of $h_{wa}$ over the width of the flume and from these measurements determine a suitable average value of $h_{wa}$ for use in (4.1). These experiments indicated that, besides varying over the flume width, $h_{wa}$ varies with air temperature, $T_a$, in the refrigerated room. The second set of experiments, for which a different apparatus was used, were conducted to verify that $h_{wa}$ varies with $T_a$, and to provide additional data. The data indicate the following relationship between $h_{wa}$ and $T_a$:

$$h_{wa} = 6.2 - 0.33T_a, \quad (B.1)$$

in which $T_a$ is in degrees Celsius. For the experiments on naleds, this relationship was used in (4.1) to determine $\phi_{wa}$. 

First Set of Experiments

A cross section of the apparatus used for the experiments is illustrated in Figure B.1. A shallow, rectangular box, constructed from insulation panels, was placed in the refrigerated flume, which was used for the naled experiments. The box was about 2 meters long, as wide as the flume, and could be filled with water to a depth of about 0.05 meters. Temperatures of the water in the box and of the overlying air were monitored using platinum resistance thermometers connected to a programmable scanner and printer. The thermometer for air temperature, \( T_a \), was hung over the center of the box about 0.6 meters above the water surface, which is approximately the same distance that the thermometers were suspended above the surfaces of the flume naleds. At the longitudinal center of the box, two thermometers measured water temperature at distances below the water surface equal to about one-third and two-thirds of the depth. To measure the distribution of \( h_{wa} \) across the width of the flume, the transverse position of these thermometers was varied from experiment to experiment. The thermometers were placed either 0.5, 0.33, 0.18, or 0.06 times the flume width from one sidewall. In total, sixteen experiments were completed.

Experimental Procedures

Before each experiment was started, the insulated box was filled to depth \( d \) with water at temperature between 4 and \( 5^\circ \)C, and the temperature of the air in the refrigerated room was set to either \(-13 \) or \(-7^\circ \)C. As the water cooled due to heat transfer to the cold air above, the water
Figure B.1. Apparatus for the first set of experiments to determine \( h_{wa} \).

and air temperatures were recorded at 5-minute intervals. An experiment was ended when ice began to form on the surface of the water.

Data Analysis

The heat-transfer coefficient \( h_{wa} \) was estimated from the data using

\[
\frac{\Phi_{wa}}{T_s - T_a} = h_{wa} = - \frac{\rho_w C_w d}{(T_s - T_a)} \frac{dT_w}{dt}, \tag{B.2}
\]
which expresses a balance between the rate of heat transfer, $\phi_{wa}$, from the water surface to the air and the rate at which heat is removed from the water. In (B.2), it is assumed that heat transfer from the water takes place only in the vertical direction and that the heat flux through the floor of the insulated box is negligible (this was verified by calculation). As described in Chapter IV and illustrated in Figure 4.1, $T_s$ is the temperature of the water surface and $T_w$ is the bulk water temperature, which for still water is equal to the depth-averaged temperature. In (B.2), $C_w$ = specific heat of water and $\rho_w$ = mass density of water. If $h_{wa}$ is assumed to be a constant with respect to time, (B.2) can be integrated from $t = t_1$ and $T_w = T_{w1}$ to $t = t_2$ and $T_w = T_{w2}$ in order to obtain

$$h_{wa} = -\frac{\rho_w C_w d (T_{w2} - T_{w1})}{\int_{t_1}^{t_2} (T_s - T_a) dt}.$$  \hspace{1cm} (B.3)

This equation was used to estimate $h_{wa}$ from the data.

In order to estimate $h_{wa}$ using (B.3) it was necessary to determine appropriate values for $T_s$ and $T_w$ from data on the water temperatures $d/3$ and $2d/3$ below the water surface. Denoting these measured water temperatures as $T_{1/3}$ and $T_{2/3}$, respectively, the expressions that were used for $T_s$ and $T_w$ are

$$T_s = T_{1/3} - K_1 (T_{2/3} - T_{1/3})$$  \hspace{1cm} (B.4)

and
\[ T_w = T_{1/3} + K_2(T_{2/3} - T_{1/3}) \]  \hspace{1cm} (B.5)

If the distribution of water temperature over \( d \) were linear, then \( K_1 \) would be 1.0 and \( K_2 \) would be 0.5. However, for the present experimental set-up an approximate analytical solution can be obtained which shows that the water temperature distribution is, in general, non-linear and also unsteady. To approximately account for these effects, the value of \( K_1 \) used was computed using (B.4) with values of \( T_{1/3} \) and \( T_{2/3} \) that were measured just before the water surface started to freeze, when \( T_s \) was known to be approximately 0\(^\circ\)C. The values of \( K_1 \) that were determined in this way were in the range 1.0 to 2.05, with most of the values less than 1.3.

For \( K_1 \) greater than 1.0, \( K_2 \) should be less than 0.5. However, for all of the experiments, it was assumed that \( K_2 = 0.5 \). Adjustments to \( K_2 \) to account for the non-linear water temperature profile were considered unjustified for three reasons: First, because it is related to a difference between two water temperatures rather than to the magnitude of a single temperature, the value determined for \( h_{wa} \) using (B.3) is relatively insensitive to small variations in \( K_2 \). Second, for values of \( K_1 \) less than 1.3, the water temperature distributions were nearly linear so that \( K_2 \) was nearly 0.5 for most of the experiments. Third, the uncertainty in \( h_{wa} \) that is due to uncertainty in the value determined for \( K_1 \) is likely larger than the improvement in the estimation of \( h_{wa} \) that can be achieved by modifying \( K_2 \).
The time values $t_1$ and $t_2$ were chosen to include the period of time over which the water cooled from an average temperature of about $3^\circ C$ to the time at which the temperature of the water surface reached $0^\circ C$. The integral in (B.3) was evaluated using the trapezoidal rule over the time period $t_1$ to $t_2$, with the time period divided into five equal intervals.

Results

The results from the first set of experiments are illustrated in Figure B.2, in which the variation of $h_{wa}$ across half of the flume's width is shown for average $T_a = -12.9$ and $-6.9^\circ C$. Figure B.2 indicates that $h_{wa}$ is smallest near the side-walls of the flume and largest in the center of the flume. To obtain average values of $h_{wa}$ for each value of $T_a$, the data on $h_{wa}$ was integrated over, and divided by, half the flume width. The trapezoidal rule was used for the approximate integration. The average values of $h_{wa}$ that were computed are given in Table B.1. A straight line was fit through the two values to obtain (B.1). This linear fit was justified by the results of the second set of experiments.

Second Set of Experiments

A vertical cross section through the apparatus used for the second set of experiments is illustrated in Figure B.3. Insulation was attached to the walls and under the bottom of a rectangular plexiglas box 0.2 meters by 0.15 meters by 0.2 meters deep. Water in the box was continually mixed by the oscillatory, vertical motion of a wooden grid. Platinum resistance thermometers were used to monitor the air tempera-
Figure B.2. Variation of $h_{wa}$ over flume's width.

Table B.1

Results from First Set of Experiments

<table>
<thead>
<tr>
<th>$T_a$, °C</th>
<th>$h_{wa}$, W/m²/°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>-12.9</td>
<td>10.4</td>
</tr>
<tr>
<td>-6.9</td>
<td>8.5</td>
</tr>
</tbody>
</table>
Figure B.3. Apparatus for the second set of experiments to determine $h_{wa}$.
ture, $T_a$; the bulk water temperature, $T_w$; and the water surface temperature, $T_s$. It was necessary to measure both $T_w$ and $T_s$ because, in spite of the mixing, the temperature near the surface of the water was somewhat lower than the temperature in the middle of the volume of water. Because of the construction of the box, which had previously been used for studies of frazil ice growth, its corners could not be adequately insulated. Consequently, in addition to the heat flux $\phi_{wa}$, a significant quantity of heat, $\phi_{ba}$, was lost through the box. The magnitude of $\phi_{ba}$ was unknown, so these experiments measured the relative variation of $\phi_{wa}$ with $T_a$, but not the absolute magnitude of $\phi_{wa}$. In total, eleven experiments were conducted using four different values of air temperature.

**Experimental Procedures**

Before each experiment was started, the box was filled to depth $d$ with water at temperature between 3 and 4°C, and the temperature of the air in the refrigerated room was set to either -5, -10, -13, or $-15^\circ C$. As the water cooled, the water and air temperatures were recorded at 5-minute intervals. An experiment was ended when the temperature of the water surface reached $0^\circ C$.

**Data Analysis**

Application of the principle of conservation of thermal energy to the mass of water, $M_w$, in the plexiglas box yields
\[ A_{ws} \phi_{wa} + A_b \phi_{ba} = - C_{mw} w \frac{dT_w}{dt}, \]  
(B.6)

in which \( A_{ws} \) = area of the water surface and \( A_b \) = total outside surface area of the box. After substituting (4.1) and expressing \( \phi_{ba} \) as equal to a heat-transfer coefficient, \( h_{ba} \), times the temperature difference \( T_w - T_a \), (B.6) may be arranged into

\[ h_{wa} = - \frac{C_{mw} w \frac{dT_w}{dt}}{A_{ws} (T_s - T_a) \frac{dt}{T_s - T_a}} - \frac{A_b h_{ba} (T - T_a)}{A_{ws} (T - T_a)}. \]  
(B.7)

It was found that the data fit closely the following relationships:

\[ T_w - T_a = C_1 e^{-\alpha t}, \]  
(B.8)

and

\[ T_s - T_a = C_2 e^{-\alpha t}, \]  
(B.9)

in which \( C_1, C_2, \) and \( \alpha \) are coefficients determined for each experiment.

Using (B.8), (B.9), and the derivative of (B.8) with respect to time, (B.7) may be reduced to

\[ h_{wa} = \frac{C_{mw}}{A_{ws}} \left( \frac{\alpha C_1}{C_2} \right) - \frac{A_b h_{ba}}{A_{ws}} \left( \frac{C_1}{C_2} \right). \]  
(B.10)

With \( \alpha, C_1, \) and \( C_2 \) determined from the data and with an assumed value for \( h_{ba} \), this expression was used to estimate \( h_{wa} \).
Because \( h_{ba} \) is unknown, the absolute magnitude of \( h_{wa} \) could not be determined from the second set of experiments. However, using an assumed value for \( h_{ba} \) of 2.45, the variation of \( h_{wa} \) with air temperature was measured. A value of 2.45 for \( h_{ba} \) results in close agreement of the data from the two separate sets of experiments.

Results

Figure B.4 shows the results from both sets of experiments. The data from the second set of experiments are scattered about the line drawn through the data from the first experiments, thus verifying that \( h_{wa} \) varies with \( T_a \) in the refrigerated room. Remarkably, the equation of the straight line that best fits the data from the second set of experiments is \( h_{wa} = 6.1 - 0.33T_a \), which matches (B.1) almost exactly. Considering the small number of data points and the scatter of the data, some credit for such close agreement must be given to coincidence. Nonetheless, the data justifies the use of (B.1) to express the relationship between \( h_{wa} \) and \( T_a \) in the refrigerated room.

Conclusions

The coefficient, \( h_{wa} \), of heat transfer from water in the flume to the air in the refrigerated room varies over the width of the flume and with the temperature of the air. Near the side-walls of the flume, \( h_{wa} \) is smaller than it is in the center of the flume because the speed at which air flows over the water is less near the walls, causing the convective heat flux to be less, and also the heat flux by radiation is less. The variation of \( h_{wa} \) with air temperature is probably due to
Figure B.4. Variation of $h_{wa}$ with air temperature in the refrigerated room.

Variations in the flow of air in the refrigerated room. Colder air temperatures require that the evaporator units operate a larger percentage of the time. The air flow that accompanies operation of the evaporator units increases the convective heat transfer from a water surface.

For the purposes of the present study, (B.1) adequately describes the relationship between $h_{wa}$ and $T_a$. 

\[ h_{wa} = 6.2 - 0.33T_a \]
BIBLIOGRAPHY


