A POLARIZED MULTI-DIMENSIONAL DISCRETE-ORDINATES RADIATIVE TRANSFER MODEL FOR REMOTE SENSING APPLICATIONS

by

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ABSTRACT

A polarized multi-dimensional radiative transfer model based on the discrete-ordinates method is developed. The model is capable of solving the monochromatic vector radiative transfer equation (VRTE) considering polarization using the four Stokes parameters and the $4 \times 4$ scattering phase matrix. The VRTE may be solved for systems containing thermal and/or collimated radiant sources as well as background sources of radiation in anisotropically scattering one-, two-, or three-dimensional Cartesian geometries. Boundary conditions can account for a variety of surfaces through the use of polarized emission vectors and bidirectional reflectance matrices.

Due to the computational complexity of this problem, the model is developed using a parallel computing paradigm. In particular, the Parallel Virtual Machine software package is used to solve for each Stokes parameter on a separate computer processing unit. Though using four processing units is optimal, the model may be run using from one to four processors.

The model is validated for one-dimensional polarized radiative transfer by comparing its results to benchmark cases available in the literature. The validation confirms that the model functions properly for plane-parallel polarized radiative transfer in situations involving thermal and collimated sources of radiation. The results are accurate so long as a quadrature set is chosen so that all phase functions used for a given problem normalize to unity. For two- and three-dimensional geometries, the model is tested by using a one-dimensional system as input and running in three-dimensional mode.

The model is applied to several example 1- and 3-D problems that illustrate the importance of accounting for edge effects in polarized radiative transfer. Two example cases consider thermal sources of radiation for 1- and 3-D microwave radiative transfer for precipitating atmospheres containing spherical and aspherical particles. A third example case considers 3-D infrared radiative transfer for a collimated source irradiating a cuboidal cloud. A final example case considers both thermal and collimated sources of radiation. Though the 3-D cases cannot be validated at the present time due to a lack of published data, the model results appear to be feasible.
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NOMENCLATURE

Roman Letter Symbols

a  electric field amplitude, V/m

$a_n$  Mie coefficient defined in Eq. (4.4)

$\overline{A}$  2 × 2 dummy submatrices used throughout the text, various dimensions

b  polarization ellipse semi-major axis lengths, various dimensions

$b_n$  Mie coefficient defined in Eq. (4.4)

c  polarization ellipse semi-minor axis length, various dimensions

$c_o$  velocity of light in a vacuum, $3 \times 10^8$ m/s

$C_{a,e,s}$  absorption, extinction, or scattering cross section, m$^2$

$C_1$  first blackbody radiation constant, $0.59544 \times 10^8$ W·μm$^4$/m$^2$

$C_2$  second blackbody radiation constant, $1.4388 \times 10^4$ μm·K

d_n  Mie coefficient defined in Eq. (4.4)

D  density, g/m$^3$

E  electric field intensity scalar, V/m

$\overline{E}$  electric field intensity vector, V/m

F  line-shape function, Hz$^{-1}$

f  frequency, Hz

$f_d$  fraction of energy that is diffuse

$f$  fractional difference, dimensionless

$f_q$  albedo (used in Chapter VII), dimensionless

g  asymmetry parameter, dimensionless

g_1, g_2  dummy variables used in Eq. (4.34)

h  Planck's constant ($6.6256 \times 10^{-34}$ J·s)

H  magnetic field intensity scalar, A/m

$\overline{H}$  magnetic field intensity vector, A/m
\( i \)  \( \sqrt{-1} \)

\( i_1, i_2 \)  polarization rotation angles, rad

\( I \)  first Stokes parameter or intensity (radiance), W/m\(^2\)-\( \mu \)m-sr

\( I_b \)  blackbody intensity, W/m\(^2\)-\( \mu \)m-sr

\( \mathbf{T} \)  Stokes vector \((I, Q, U, V)^T\), W/m\(^2\)-\( \mu \)m-sr

\( k \)  wavenumber \((= 2\pi/\lambda)\), rad/m

\( k_a \)  absorption coefficient, Np/m

\( \mathbf{k}_a \)  \( 1 \times 4 \) absorption vector, Np/m

\( k_e \)  extinction coefficient, Np/m

\( \mathbf{k}_e \)  \( 4 \times 4 \) extinction matrix, Np/m

\( k_s \)  scattering coefficient, Np/m

\( m \)  complex refractive index, \( m = m_r + m_i \)

\( \mathbf{L} \)  \( 4 \times 4 \) polarization rotation matrix, dimensionless

\( \mathbf{M} \)  \( 4 \times 4 \) Mueller matrix, dimensionless

\( M_{\alpha \beta} \equiv 2\pi i/k \left< N(r, \Xi) S_{\alpha \beta} \right> \)

\( N \)  particle size distribution, 1/volume/radius

\( P \)  pressure, mbar

\( \mathbf{P} \)  \( 4 \times 4 \) scattering phase matrix, Np/m

\( Q \)  second Stokes parameter, W/m\(^2\)-\( \mu \)m-sr

\( Q_{a,e,s} \)  absorption, extinction, or scattering efficiency factor, dimensionless

\( q \)  radiant flux, W/m\(^2\) or W/m\(^2\)-\( \mu \)m

\( R \)  rainrate, mm/hr

\( R_h \)  horizontal reflection coefficient, dimensionless

\( R_v \)  vertical reflection coefficient, dimensionless

\( \mathbf{R} \)  \( 4 \times 4 \) bidirectional reflection matrix, dimensionless

\( r \)  scattering distance or particle radius, various length dimensions

\( s \)  distance along line-of-sight direction, various dimensions
$\mathbf{S}$  Poynting vector, W/m²

$S$  scattering amplitude functions: $S_1, S_2, S_3, S_4$, dimensionless

$S_{\gamma\beta}$  $S_{\gamma\gamma} \equiv S_2, S_{\gamma\beta} \equiv S_3, S_{\beta\gamma} \equiv S_4, S_{\beta\beta} \equiv S_1$

$t$  time, s, or transmittance, dimensionless

$\mathbf{T}$  $4 \times 4$ transmission matrix, dimensionless

$T$  temperature, K

$T_B$  brightness temperature, K

$U$  third Stokes parameter, W/m²·μm·sr

$V$  fourth Stokes parameter, W/m²·μm·sr

$\mathbf{W}$  $4 \times 4$ basis transformation matrix, dimensionless

$X$  size parameter ($X = 2\pi r_m / \lambda = kr$)

$x$  spatial coordinate, various dimensions

$Y = m X$

$y$  spatial coordinate, various dimensions

$Z$  degree of polarization, dimensionless

$\mathbf{Z}$  rotated $4 \times 4$ scattering phase matrix, Np/m

$z$  spatial coordinate, various dimensions
Greek Letter Symbols

$\alpha$  finite-difference weighting factor, dimensionless
$\beta$  ellipticity, rad
$\chi$  orientation angle, rad
$\delta$  phase of wave, rad
$\epsilon$  emittance, dimensionless
$\overline{\epsilon}$  $1 \times 4$ emission vector, dimensionless
$E$  internal energy, eV ($1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$)
$\Phi$  scattering phase function, dimensionless
$\overline{\Phi}$  $4 \times 4$ normalized Mueller matrix, dimensionless
$\phi$  azimuthal angle, rad or deg
$\gamma$  generic radiative property, m$^{-1}$
$\overline{\theta}$  identity matrix
$\eta$  intrinsic impedance, ohms
$\lambda$  wavelength, m or $\mu$m
$\mu$  direction cosine ($= \cos \theta$)
$\nu$  cosine of angle between propagation direction and normal to boundary
$\pi$  3.14159 …
$\Pi_n$  Mie function defined in Eq. (4.6)
$\pi_n$  scattering angle-dependent Mie coefficient defined in Eq. (4.10)
$\rho$  reflectance, dimensionless
$\Sigma$  line-strength, Hz
$\Theta$  scattering angle, rad
$\theta$  polar angle, rad or deg
$\tau$  optical thickness, dimensionless
$\tau_n$  scattering angle-dependent Mie coefficient defined in Eq. (4.10)
$\Omega$  solid angle, sr
<table>
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<tr>
<td>( \omega )</td>
<td>circular frequency (= ( kc )), rad/s</td>
</tr>
<tr>
<td>( \varpi )</td>
<td>single scattering albedo, dimensionless</td>
</tr>
<tr>
<td>( \xi )</td>
<td>phase variable (= ( kz - \omega t )), rad</td>
</tr>
<tr>
<td>( \Psi_n )</td>
<td>Mie function defined in Eq. (4.6)</td>
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<tr>
<td>( \Xi )</td>
<td>particle orientation</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>dummy angle used in Eq. (3.29), rad</td>
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<tr>
<td>( \Upsilon_n )</td>
<td>Mie function defined in Eq. (4.7)</td>
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List of Subscripts

ba background

c cloud or collimated

e destination (end)

g gas

h horizontal

i incident (also used as dummy index in Chapter 4)

I, Q, U, V refers to the given Stokes parameter

j, k dummy indices

l low energy state

m high energy state

n dummy index

o boundary

p precipitation

r reference

s scattered

t transmitted

v vertical or vapor

x, y, z coordinate axis

\( \lambda \) spectral quantity

* refers to alternate Stokes vector \( \mathbf{T}_* = (I_v, I_h, U, V)^T \)
**List of Superscripts**

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<tr>
<td>o</td>
<td>from outside boundary</td>
</tr>
<tr>
<td>p</td>
<td>cell-mean</td>
</tr>
<tr>
<td>T</td>
<td>matrix transform operator</td>
</tr>
<tr>
<td>'</td>
<td>incoming quantity</td>
</tr>
<tr>
<td>*</td>
<td>complex conjugate</td>
</tr>
<tr>
<td>-</td>
<td>moving away from boundary</td>
</tr>
<tr>
<td>+</td>
<td>moving toward boundary</td>
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**List of Special Symbols**

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<td>(\Delta x, \Delta y, \Delta z)</td>
<td>differential lengths in the x, y, and z directions, various dimensions of length</td>
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<tr>
<td>(\Delta V)</td>
<td>differential volume (\Delta x\Delta y\Delta z), various dimensions of volume</td>
</tr>
<tr>
<td>~</td>
<td>modified quantity</td>
</tr>
<tr>
<td>—</td>
<td>vector quantity</td>
</tr>
<tr>
<td>(\equiv)</td>
<td>matrix quantity</td>
</tr>
<tr>
<td>(\nabla)</td>
<td>gradient vector operator</td>
</tr>
<tr>
<td>(&lt;&gt;)</td>
<td>time-averaging or size-distribution averaging operator</td>
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<tr>
<td>Acronym</td>
<td>Description</td>
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<tr>
<td>AMMS</td>
<td>Advanced Microwave Moisture Sounder</td>
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<td>AVHRR</td>
<td>Advanced Very High Resolution Radiometer</td>
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<td>CDR</td>
<td>Circular Depolarization Ratio</td>
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<td>DDA</td>
<td>Discrete Dipole Approximation</td>
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<td>DMSP</td>
<td>Defense Meteorological Satellite Program</td>
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<td>DRF</td>
<td>Divergence of Radiative Flux</td>
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<td>Discrete-Ordinates Method</td>
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</tr>
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<td>EOS</td>
<td>Earth Observing System</td>
</tr>
<tr>
<td>FOV</td>
<td>Field of View</td>
</tr>
<tr>
<td>GPCP</td>
<td>Global Precipitation Climatology Project</td>
</tr>
<tr>
<td>GSFC</td>
<td>Goddard Space Flight Center</td>
</tr>
<tr>
<td>HMMR</td>
<td>High-resolution Multi-frequency Microwave Radiometer</td>
</tr>
<tr>
<td>IFOV</td>
<td>Instantaneous Field of View</td>
</tr>
<tr>
<td>LDR</td>
<td>Linear Depolarization Ratio</td>
</tr>
<tr>
<td>MPI</td>
<td>Message Passing Interface</td>
</tr>
<tr>
<td>MSU</td>
<td>Microwave Sounding Unit</td>
</tr>
<tr>
<td>MW</td>
<td>Microwave</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
</tr>
<tr>
<td>PVM</td>
<td>Parallel Virtual Machine</td>
</tr>
<tr>
<td>RTE</td>
<td>Radiative Transfer Equation</td>
</tr>
<tr>
<td>SMMR</td>
<td>Scanning Multichannel Microwave Radiometer</td>
</tr>
<tr>
<td>SSM/I</td>
<td>Special Sensor Microwave/Imager</td>
</tr>
<tr>
<td>$T_B$</td>
<td>Brightness Temperature</td>
</tr>
<tr>
<td>TIROS</td>
<td>Television Infrared Observation Satellite</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>TRMM</td>
<td>Tropical Rainfall Measuring Mission</td>
</tr>
<tr>
<td>VIS</td>
<td>Visible</td>
</tr>
<tr>
<td>VDOM</td>
<td>Vector Discrete Ordinates Method</td>
</tr>
<tr>
<td>VRTE</td>
<td>Vector Radiative Transfer Equation</td>
</tr>
<tr>
<td>WCRP</td>
<td>World Climate Research Program</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

1.1 Background

The understanding of global climate patterns and fluctuations has become a central issue in environmental, political, and scientific communities within the last several years. To begin to understand the impact that human activities and nature have upon weather patterns, it is important to establish methods of observing and measuring various meteorological quantities, particularly rainfall. Although ground-based radar and raingauge networks provide means of measuring rainfall amounts, these measurement systems are difficult to employ on a global scale. Because approximately seventy percent of the surface of the Earth is covered by water, ground-based measurement systems are inadequate for obtaining precipitation rates over the oceans. In these situations, satellite based remote sensing methods are particularly vital for the accurate and long-term monitoring of precipitation events.

Over the last decade, several projects dedicated to the accurate measurement of global precipitation have been developed. For example, the Special Sensor Microwave/Imager (SSM/I) was launched on June 19, 1987 as part of the Defense Meteorological Satellite Program (DMSP) (Hollinger et al., 1990). Another project, the Global Precipitation Climatology Project (GPCP) established by the World Climate Research Program (WCRP), was created to produce monthly global rainfall totals on a 2.5° latitude by 2.5° longitude scale for the period 1986-95 using data returned from satellite observations. In addition, NASA's Earth Observing System (EOS) includes a High-resolution Multi-frequency Microwave Radiometer (HMMR) that is used for remote sensing purposes. Finally, it is anticipated that in August 1997 a satellite will be launched as part of the Tropical Rainfall Measuring Mission (TRMM), a program being developed jointly by Japan and the United States (Simpson, 1988; Simpson et al., 1988). Other programs have developed precipitation retrieval algorithms based on measurements obtained from the Nimbus-5 Electrically Scanning Microwave Radiometer (ESMR), the Nimbus-7 Scanning Multichannel Microwave Radiometer (SMMR), and the Microwave Sounding Unit (MSU) aboard the TIROS-N series of weather satellites.

Many methods have been developed that use visible (VIS) and infrared (IR) satellite observation to estimate precipitation amounts (Barrett and Martin, 1981; Barrett, 1993). However, as is discussed later, at these frequencies, the radiant energy observed by
satellite radiometers originates from cloud tops and, therefore, is an indirect observation of the precipitation below. In contrast, microwave (MW) measurements provide a more direct means of observing and estimating precipitation, as is discussed later. Therefore, the research proposed here focuses on the MW portion of the electromagnetic spectrum rather than VIS or IR. Also, it should be emphasized that the research reported in this document focuses on passive remote sensing applications. This is in contrast to active remote sensing schemes, for example, radar, in which a signal is beamed into the atmosphere and the returning echo is measured by a receiver. In passive remote sensing schemes, a receiver measures radiant energy that is naturally emitted by some source, typically the surface and atmosphere of the Earth.

The basic principles of passive microwave measurement of precipitation are reviewed by considering the remote sensing scenario shown in Fig. 1.1. In this situation, a downward viewing radiometer onboard a satellite measures radiant energy that originates through emission from the surface and atmosphere of the Earth, as well as scattered solar energy and cosmic background radiation. This energy is subject to redistribution through interactions (absorption and scattering) with various atmospheric constituents; for example, clouds, rain, ice, and atmospheric gases and aerosols. In addition, the surface of the Earth reflects a certain portion of energy incident upon it (this fraction is known as the albedo), and absorbs the remaining fraction. The signal that reaches a satellite is a function of several variables, including the frequency (or wavelength) of the energy being observed, and the amount and type of matter through which the radiant energy must propagate, typically described by a parameter called the optical thickness. Factors, such as the spatial arrangement of this matter, the radiometer viewing angle and antenna pattern, the physical attributes of the atmosphere (temperature, pressure, density, etc.), and the composition and topology of the underlying surface, contribute to the radiant energy pattern that reaches the satellite.
Figure 1.1 Satellite remote sensing scenario.

The mathematical description of the emission, absorption, and scattering processes described above is given by the radiative transfer equation (RTE). By solving the RTE, the radiant energy that a satellite would "see" under various conditions can be estimated. This energy is typically described in terms of a blackbody equivalent temperature known as a brightness temperature \( T_B \), which is described more fully by Ulaby et al. (1981). As an example, consider Fig. 1.2, which is an adaptation of a \( T_B \)-rainrate relationship first shown by Wilheit et al. (1977). These curves show how the energy received by a satellite sensor over an ocean surface varies with rainrate and freezing level at the microwave frequency of 19 GHz. For instance, the figure shows that, for a freezing level of 4 km and a rainrate of 10 mm/hr, the scene "appears" to the satellite to have a temperature of about 250 K. This type of simulation is sometimes called a "forward problem." A more difficult problem is the "inverse problem," where a satellite might indicate a brightness temperature of 250 K, and the corresponding rainrate must be found. As can be seen even for the case shown in Fig. 1.2, this relationship may not be unique but could be double-valued. In addition, the curves shown in Fig. 1.2 presuppose a particular atmospheric structure. In reality, the details of this structure are usually unknown. For a more detailed introduction to precipitation measurement by satellites, the reader is referred to Wilheit (1986) and Barrett (1993).
1.2 Motivation

Current satellite receivers (e.g. SSM/I, ESMR, MSU) as well as future satellite missions (e.g. TRMM) have instruments that observe polarized radiant energy quantities. As is discussed in detail later, the complete polarization state of an electromagnetic wave at a particular frequency is described by the four Stokes parameters, usually labeled I, Q, U, and V. Most of the instruments mentioned above measure two polarized radiant energy quantities for a given frequency: the magnitude of vertically and horizontally polarized radiation, which are linear combinations of two of the four Stokes parameters. However, the theory of "polarimetry" measures brightness temperatures corresponding to all four Stokes parameters. Polarimetry has typically been applied to the passive remote sensing of wind direction (Johnson, et al., 1993) and radar remote sensing of rainfall (Vivekanandan et al. 1993) but it appears that there are very few applications of polarimetry to passive remote sensing of rainfall.

As an example of a case where all four Stokes parameters are important, the plots shown in Fig. 1.3 demonstrate that even the assumption of a two-parameter Stokes vector may lead to erroneous results. The results are from a model designed by Tsang et al. (1985) that considers thermal emission of small, horizontally oriented, oblate spheroids in a plane-parallel slab with a thickness of 100 meters. The volume fraction is 0.004, and the lengths of the semi-major and -minor axis of the spheroids are 1 and 0.05 cm, respectively. The particles have a dielectric constant of $(30.68 + 1.55i)$ times that of
Figure 1.3 Comparison between two Stokes parameter and four Stokes parameter results. See text for explanation (adapted from Tsang et al., 1985).

free space, and a uniform temperature of 300 K. The results are for a frequency of 1 GHz, and scattering is ignored. The results show that the two- and four-parameter results are completely different for zenith viewing angles less than about forty-five degrees.

To account for polarization, several researchers have developed models to solve the vector radiative transfer equation (VRTE), particularly in the astrophysics and cloud remote sensing communities. However, these VRTE models are limited to solving plane-parallel problems, and have seldom been applied to remote sensing of precipitation problems, even though polarization signatures have been observed in microwave remote sensing of rainfall. For example, Spencer (1989) observed 85 GHz polarization differences up to about 10 K in convective rainfall cores over land, and suggested that the differences are the result of oriented ice crystals. At lower frequencies (18 and 37 GHz), Adler et al. (1990) found the mean polarization difference in precipitation areas to be about 6 K, which was attributed to nonspherical horizontally oriented ice particles.
1.3 Objectives of Research

A radiative transfer model based on the discrete-ordinates method (DOM) was developed by Sánchez (1991). This work is a general purpose radiative transfer solver that has been applied to multi-dimensional systems for remote sensing as well as engineering heat transfer problems (Sánchez et al., 1992). The model uses the DOM to solve the unpolarized RTE for one-, two-, or three-dimensional Cartesian geometries under a wide variety of boundary conditions, accounting for spectral and spatial variations of non-homogeneous boundaries, and provides output intensities in any desired direction. The original model has proven itself to be very flexible, accurate, and computationally efficient. However, it lacks the ability to solve the polarized (vector) form of the transfer equation. The addition of this feature to the model would make it a more comprehensive solver of radiative transfer problems.

In the microwave portion of the spectrum, where remotely sensed observations of rainfall are made, atmospheric polarization effects are caused by the underlying surface as well as suspended hydrometeors. Many existing rainfall remote sensing models neglect polarization altogether, and others neglect the contribution of hydrometeors and consider surface polarization effects only. Those that do solve the VRTE to account for surface and hydrometeor contributions to polarization consider plane-parallel (i.e. 1-D) radiative transfer only. Thus, the primary objective of this research is to extend the formulation of the multi-dimensional DOM to solve the VRTE, and to test this formulation on various atmospheric scenarios. In addition, the model has applications for the engineering heat transfer community, particularly in laser diagnostic and particle sizing types of studies (Look et al., 1992).

In Section 1.3.1, the objectives of enhancing the DOM model to account for polarization using the Stokes vector approach are discussed. Section 1.3.2 considers possible applications of the enhanced model, and Section 1.3.3 describes the limitations of the model.

1.3.1 Objectives of Model Development

The scalar version of the DOM is capable of solving the one-, two-, and three-dimensional scalar RTE for inhomogeneous, anisotropically scattering media under a variety of boundary conditions. The model has been validated and tested by comparing its results to those of other studies, as documented by Sánchez et al. (1992). In addition, the model has been applied to satellite remote sensing simulations of precipitating atmospheres, as reviewed in Chapter II. An important objective of the present research is to enhance the
DOM by extending it to include polarization using the Stokes parameter formalism. Specifically, the objectives of the model development are:

1. The existing DOM model is to be extended to include polarized radiative transfer using the Stokes vector approach. The model should continue to be applicable to one-, two-, and three-dimensional situations.

2. The model must be capable of interfacing with existing models and databases of single scattering properties for aspherical particles. The single scattering properties include the scattering amplitude functions, defined in Chapter III. These functions are used to compute the phase and extinction matrices, and the absorption vector, described in Chapters III and IV.

3. The model is to take advantage of parallel and distributed computing paradigms, as outlined in Chapter V. The model should run on scalar, distributed, or parallel computer architectures.

4. The model must be tested and validated against existing results.

1.3.2 Objective of Model Application

The main impetus behind the development of the multi-dimensional polarized radiative transfer model is the need to more accurately simulate microwave radiative transfer in three-dimensional precipitating atmospheres. Thus, a secondary objective of the proposed research is to apply the model to precipitating systems, to demonstrate possible differences between three-dimensional polarized radiative transfer and other types of radiative transfer modeling. The results of the numerical experiments should demonstrate the effects of the inclusion of polarized radiative transfer for both plane-parallel and finite precipitating systems.

1.3.3 Limitations of Model

The objectives outlined above for the polarized DOM are very ambitious, and may give the impression that the model is a complete radiative transfer solver. Nevertheless, the model has certain limitations. For example, the model does not:

1. solve time-dependent radiative transfer;

2. account for polarized background radiation (though unpolarized is considered);
3. solve thermal radiative equilibrium problems that require coupling with other modes of heat transfer;

4. incorporate the techniques used by Sánchez (1991) for a) relaxation (to improve convergence); b) inclusion of solids; and c) δ-M and modified δ-M scaling (Kim and Lee, 1990) of the phase function.

Otherwise, the model retains all the features of the scalar DOM of Sánchez (1991).

1.4 Implications and Contributions

As emphasized throughout this Chapter, most VRTE models use a plane-parallel assumption. The development of a multi-dimensional polarized radiative transfer model as described in this Chapter is a unique contribution to the field of radiative transfer. In particular, the development of such a model is a unique contribution because 1) it appears that it is currently the only existing model capable of solving the full VRTE for 3-D Cartesian geometries including thermal and direct sources of radiation; 2) it extends the DOM technique that has been previously applied only to scalar RTE problems to the vector DOM that solves the VRTE; and 3) it demonstrates how parallel computing techniques can be applied to the solution of the VRTE. Such a model should, for example, be useful to researchers interested in microwave remote sensing, and, more generally, should add to the understanding of polarized radiative transfer.

1.5 Outline of Contents

Chapter II contains a literature survey of several topics related to the research. Among the topics reviewed are a history of passive microwave remote sensing of precipitation, multi-dimensional radiative transfer algorithms, and polarization, including polarized radiative transfer and polarized single scattering theory. Chapter III discusses the topic of polarized radiative transfer, and introduces the vector radiative transfer equation. In Chapter IV, the methods for computing the radiative properties of hydrometeors, atmospheric gases, and surfaces are reviewed. Chapter V presents the DOM solution procedure, and also includes an overview of parallel and distributed computing aspects of this solution procedure. A validation of the model is given in Chapter VI, and Chapter VII presents examples of new capabilities of the model. Conclusions and recommendations are given in Chapter VIII. Finally, the work is supplemented by several appendices and a list of references.
CHAPTER II
LITERATURE REVIEW

In this chapter, several works related to the topics of remote sensing and polarization are reviewed. In Section 2.1, a review of many of the important papers and books that cover the topic of passive microwave remote sensing of precipitation is presented. In Section 2.2 a literature survey for multi-dimensional radiative transfer is given. Section 2.3 lists several works that deal with the general issues of scattering and polarization, and finally, Section 2.4 covers the specific topic of polarized radiative transfer.

2.1 Passive Microwave Remote Sensing of Precipitation

Textbooks that cover the topic of microwave remote sensing have been prepared by Ulaby et al. (1981), Tsang et al. (1985), and Janssen (1993). A book that deals specifically with the measurement of precipitation using satellite remote sensing is that of Barrett and Martin (1981). The first model that directly relates upwelling radiances to precipitation amounts is that of Wilheit et al. (1977), who shows functional relationships between rainfall rates and upwelling brightness temperatures at a wavelength of 1.55 cm for a precipitating cloud over an ocean. This model provides the framework that the majority of subsequent passive microwave retrieval studies are based upon (Huang and Liou 1983; Spencer et al. 1983, 1989; Liu and Curry, 1992; Haferman et al., 1993a; Chang et al., 1993). In general, these studies show that, in the microwave region, brightness temperatures increase with increasing rainrate over the ocean surface. This is because the ocean has a low emittance in the microwave region and appears radiometrically cold, while precipitating cloud particles appear relatively warm. At a certain rainfall rate, however, the brightness temperature becomes saturated and then decreases. On the other hand, land has a relatively high emittance and, thus, looks radiometrically warm, so that brightness temperatures generally decrease with increasing rainrate.

The importance of scattering effects by precipitation on observed radiometric quantities is elucidated by Wilheit et al. (1982) and Spencer et al. (1983). These studies demonstrate that, at lower frequencies (19.35 GHz), precipitation causes brightness temperatures to increase over cold backgrounds, while, at higher frequencies (87.5, 92 GHz), scattering effects cause brightness temperatures to decrease over warm backgrounds, particularly for hydrometeors above the freezing level. In addition, the atmosphere is generally more transparent at lower frequencies, which provide better direct
estimates of surface conditions, whereas the higher frequencies are better estimators of conditions at higher altitudes.

Mugnai and Smith (Mugnai and Smith 1988; Smith and Mugnai, 1988; Smith and Mugnai, 1989; Mugnai et al., 1990; Smith et al., 1992; Mugnai et al., 1993) use two- and three-dimensional cloud models in combination with emission and scattering weighting functions to describe where radiation originates and how it ultimately reaches the satellite. Adler et al. (1991) point to the utility and necessity of using numerical cloud models for microwave radiative transfer modeling in the atmosphere. Though these studies use multidimensional cloud models to provide realistic vertical and horizontal distributions of hydrometeors, the radiative transfer equation is solved using a plane-parallel assumption.

2.2 Multi-Dimensional Radiative Transfer

Due to the multi-dimensional spatial nature of precipitating systems, the plane-parallel radiative transfer approach used in most satellite remote sensing models cannot fully account for the complexity inherent in such systems. Though many studies have acknowledged this shortcoming, few have explored it in detail. Weinman and Davies (1978) use an analytical model based on the Eddington approximation and a Monte Carlo model to show that microwave radiances from horizontally finite clouds of hydrometeors are significantly less than those from plane-parallel clouds. Kummerow (Kummerow, 1987; Kummerow and Weinman, 1988b; Weinman and Kummerow, 1988) investigate finite horizontal effects in a two-layer precipitating atmosphere with oblate spherical particles using the Eddington second approximation to solve the equation of transfer. The results show that large errors can occur as a result of finite effects. Recognizing these findings, Olson (1989) developed a retrieval algorithm based on the finite radiative transfer model of Weinman and Davies (1978). A related problem is the horizontal field-of-view or "beam-filling" problem (Lovejoy and Austin, 1980; Chiu et al., 1990; Short and North, 1990; Turner and Austin, 1993), which is a result of non-uniform rainrates within a radiometer field-of-view.

Stephens (1989) suggests that the effects of horizontal inhomogeneity are most prevalent in media dominated by scattering, and less of a concern for media dominated by absorption and emission, for example, media at low microwave frequencies. The study by Haferman et al. (1994a) demonstrates that, for a frequency of 19.35 GHz, finite effects can be highly prevalent at intermediate rainrates. In that study, the atmospheric model of Wilheit et al. (1977) is incorporated into a three-dimensional setting and upwelling brightness temperatures are computed using a three-dimensional radiative transfer model based on the discrete-ordinates method (Sánchez et al., 1992).
2.3 Polarization and Scattering

Textbooks that discuss fundamental concepts of polarization, scattering, and wave propagation include van de Hulst (1980; 1981), Bohren and Huffman (1983), Coulson (1988), and Ishimaru (1978; 1991). Textbooks that discuss the topic of scattering from rough surfaces are the classic work of Beckmann and Spizzichino (1963), and Ogilvy (1991). Computer programs to compute quantities associated with scattered electromagnetic fields are provided for spherical particles, coated spherical particles, and infinite cylinders by Bohren and Huffman (1983), and for slabs, infinite cylinders, spheres, and axisymmetric aspherical particles by Barber and Hill (1990). Such quantities are needed to compute the radiative coefficients needed by the RTE, and the vector and matrix radiative coefficients used in the VRTE. Additional relevant works that discuss scattering theory include Oguchi (1981), Olsen (1982), Bringi et al. (1982), Mon (1982), Kummerow and Weinman (1988a), and Kiriakie and Krajewski (1994). Papers discussing fundamental radiometric quantities and measurement methods include Doyle (1985), Bickel and Bailey (1985), Bickel et al. (1987), Freund (1990), Anderson (1991), Zimmermann and Dalcher (1991), and Kostinski et al. (1993).

In general, radiative properties used in the RTE and VRTE are obtained by solving the electromagnetic wave equations. For spheres, an exact analytical solution is possible, and is generally attributed to Mie (1908); however, Bohren and Huffman (1983) note that it is more likely that Lorenz was the first to construct such a solution. Nevertheless, the theory of scattering of electromagnetic waves by spherical particles is typically referred to as the Mie theory. Analytical solutions to the electromagnetic wave equations are available for other types of particles, for example, infinite cylinders and coated spheres (Bohren and Huffman, 1983), but in general, numerical solution techniques must be used.

Various numerical solution techniques to the electromagnetic scattering problem are available and, in this work, two such techniques are used: the discrete dipole approximation (DDA; Purcell and Pennypacker, 1973) and the T-matrix approximation (Waterman, 1971). Thus, the remaining discussion in this section reviews these two methods. For a list of other computational methods for scattering problems, the reader is referred to Flatau1 and Wriedt2.

For the DDA, an arbitrary target is replaced by an array of point dipoles with associated polarizabilities. For the original arbitrary target, a solution of the

1 "SCATLIB Light Scattering Codes" on the Internet at URL http://atol.ucsd.edu/~pflatau/index.html

2 "Electromagnetic SCATTERING Codes List" on the Internet at URL http://imperator.cip-iwl uni-bremen.de/fig01/codes2.html
electromagnetic scattering problem may be expressed in terms of an integral over the target volume of a Green function of the target. By replacing the target volume with discrete dipoles, the Green function integral is effectively discretized, resulting in a system of coupled linear equations, the solution of which allows the solution to the original electromagnetic wave equation. In theory, no restrictions exist on the target geometry, thus allowing electromagnetic scattering solutions for virtually any particle shape. However, in practice, the accuracy of the solution is dictated by the number of dipoles used, the spacing between dipoles, and the magnitude of the complex refractive index. Further details on the method, including its history, mathematical formulation, and accuracy are given by Draine and Flatau (1994). Improvements to the numerical implementation of the DDA are discussed by Flatau et al. (1990) and Goodman et al. (1990).

The T-matrix method is another approximate numerical method that, in theory, can provide solutions to the electromagnetic scattering problem for arbitrary particle geometries. However, in practice, it has been applied mostly to axisymmetric particles (Barber and Hill, 1990; Mugnai and Wiscombe, 1986, 1989; Wiscombe and Mugnai, 1986, 1988). The approach involves first expanding the electric fields in terms of vector spherical harmonic functions, and then numerically integrating functions of the expansion coefficients over the particle surface. The solutions to these integrals for the particle incident and internal fields are combined to yield a transition- or T-matrix that describes the conversion of the incident electromagnetic field into the scattered field. Mishchenko (1991) has devised an analytical averaging technique that allows rapid computation of polydispersions of axisymmetric particles using the T-matrix method. The method yields accurate results for particles with large size parameters and aspect ratios (Mishchenko and Travis, 1994a). Benchmark scattering results are presented by Mishchenko (1993), Kuik et al. (1992), Wauben et al. (1993), and Mishchenko and Travis (1994b).

### 2.4 Polarized Radiative Transfer

The formulation of the equation of transfer for polarized radiation in a plane-parallel Rayleigh scattering atmosphere is presented by Chandrasekhar (1960), and is also reviewed by Plass et al. (1973). An overview of the theory and computational aspects for polarized, anisotropically scattering atmospheres is given by Hansen and Travis (1974), who use an adding/doubling method to solve the polarized equation of transfer. More detailed forms of the equation of transfer are developed in a series of papers by Ishimaru (Ishimaru and Cheung, 1980a; Ishimaru and Cheung, 1980b; Cheung and Ishimaru, 1982; Ishimaru et al., 1982; Yeh et al., 1982; Ishimaru and Yeh, 1984; Ishimaru et al., 1984a, 1984b; Lam and Ishimaru, 1993), Tsang (Tsang et al., 1977; Tsang and Kong, 1978;
Tsang et al., 1981; Tsang 1984; Tsang et al., 1984; Tsang and Chen, 1990; Tsang and Ding, 1991) and Liou (Huang and Liou, 1983; Takano and Liou, 1993). Other papers of interest include Hovenier (1969; 1971), Ito and Oguchi (1989), and Kuga (1991). Textbooks that discuss the VRTE include Ishimaru (1978; 1991), Liou (1980; 1992), Tsang et al. (1985), and Janssen (1993). The motivation behind these works varies considerably: for example, the work of Hansen and Hovenier (1974) attempts to infer the composition of planetary atmospheres (particularly Venus) using polarized radiative transfer theory, while the studies of Ishimaru are more motivated by telecommunication concerns, where it is important to understand how polarized transmitted signals are affected by suspended atmospheric hydrometeors.

Examples of polarized radiative transfer studies include that of Ishimaru and Cheung (1980a), who consider polarized radiative transfer through spherical rain droplets. The formulation is fairly general, but, because of the assumption of spherical particles, the first two Stokes parameters (I and Q) are decoupled from the third and fourth Stokes parameters (U and V). This is an example of a study that is motivated by a need to improve the understanding of the effect of hydrometeors on telecommunication links.

Huang and Liou (1983) consider polarized radiative transfer through a precipitating atmosphere at frequencies of 19.35, 37.0, and 85.5 GHz. This is perhaps the first study concerned with remote sensing of precipitation that considers the vector form of the radiative transfer equation. They demonstrate the influence of the scattering phase function and non-isothermal cloud structure on upwelling brightness temperatures at these frequencies. In addition, they show that polarization effects can be significant over water, but that these effects tend to diminish with increasing rainrate and/or increasing frequency. As in the Ishimaru and Cheung (1980a) study, spherical particles are assumed, and thus, only the first and second Stokes parameters are considered.

Wu and Weinman (1984) solve the radiative transfer equation using polarized radiative coefficients for aspherical particles, but treat the vertically and horizontally polarized intensities as being uncoupled. In other words, they solve the scalar radiative transfer equation twice: once for the horizontally polarized intensity, and once for the vertically polarized intensity. Bauer and Schluessel (1993) consider the VRTE in their rainfall retrieval scheme, but assume spherical particles and thus treat the extinction coefficient as a scalar quantity. Takano and Liou (1993) use horizontally oriented hexagonal ice crystals in their VRTE study, but assume unpolarized surface emission in a plane-parallel atmosphere so that the third and fourth Stokes parameters are zero. They (incorrectly) treat the extinction coefficient as a scalar quantity, but later demonstrate that this assumption creates negligible error (Takano and Liou, 1994).
Many papers dealing with vector radiative transfer have been written by Evans. For example, Evans and Vivekanandan (1990) perform a study in which multiparameter radar and microwave radiative transfer models are developed to study the radar and radiometric properties of nonspherical oriented atmospheric ice particles. The VRTE is used, but because thermal emission is assumed as the only source of radiation in a plane-parallel geometry, only the I and Q Stokes parameters are nonzero. This restriction is relaxed by Evans and Stephens (1990; 1991), who formulate a multi-stream doubling-adding formulation of the VRTE and consider all four Stokes parameters for media containing randomly oriented particles with a plane of symmetry. All of these studies have implications for remote sensing of precipitation, as do the more recent studies of Evans and Stephens (1993; 1995b).

Many fundamental studies of polarized radiative transfer have been performed by Mishchenko. For example, Mishchenko (1990a) provides a general formulation of the VRTE. Mishchenko (1990b) presents the theory for solving the VRTE for partially aligned interstellar particles, and gives several computational results. A computationally efficient method for solving the VRTE for Rayleigh scattering situations is given by Mishchenko (1990c). Polarization through raindrops is considered by Mishchenko (1992), who solves electromagnetic field equations rather than the VRTE to compute polarized radiant energy quantities. Errors induced by the neglect of polarization in radiative transfer computations are discussed by Mishchenko (1994a).

Other applications of polarized radiative transfer are presented by Weinman and Guetter (1977), Spencer (1988), Hinton and Olson (1988), Kattawar and Adams (1989), Weng (1992a; 1992b), Vermote and Tanré (1992), Jin (1992), and Adams and Kattawar (1993). In some of these models, polarization effects are due to surface phenomenon only, and the scalar RTE is solved using polarized emittances and reflectances, while in other models, the plane-parallel VRTE is solved. A study of two-dimensional, polarized radiative transfer is presented by Jin (1991), though only two of the four Stokes parameters are non-zero because spherical particles are considered. Benchmark results for the VRTE are given by Evans and Stephens (1991), Wauben and Hovenier (1992), Wauben et al. (1993), and Wauben, et al. (1994).

2.5 Summary

In this chapter, many important books and papers relevant to the topics of remote sensing and polarization have been reviewed. Some references on the general topic of microwave remote sensing were provided, and, a review of the specific topic of passive microwave remote sensing of precipitation was presented. Literature related to the study of
multi-dimensional radiative transfer, particularly in atmospheric systems, was reviewed. Books and papers covering the topic of single particle scattering and polarization were listed. Finally, some background literature dealing with the topic of polarized radiative transfer was presented.
CHAPTER III
RADIATIVE TRANSFER AND POLARIZATION

3.1 Introduction

This section introduces the topic of polarization. To discuss this topic, some background in electromagnetic theory (Cheng, 1989) is necessary. In this introduction, only a few of the basic ideas are discussed. Consider Fig. 3.1 which depicts a planar transverse electromagnetic wave (that is, it has only x- and y- components) propagating through some medium. This plane wave is characterized by the orthogonal electric and magnetic vectors $\mathbf{E}$ and $\mathbf{H}$, each perpendicular to the direction of propagation. The flow of energy and wave propagation direction are described by the Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$ (3.1)

Because $\mathbf{E}$ and $\mathbf{H}$ are related, the following discussion applies to both vectors, however, it is customary to use $\mathbf{E}$ for discussion purposes. The inset in Fig. 3.1 shows how $\mathbf{E}$ is decomposed into orthogonal components $\mathbf{E}_v$ and $\mathbf{E}_h$, where subscript $v$ stands for "vertical" referring to the axes parallel to a plane of reference through the direction of propagation, and subscript $h$ stands for "horizontal" representing the axes perpendicular to this same reference plane. The selection of the plane of reference is arbitrary, though for scattering problems, it is customarily taken as the plane containing the incident and scattered beams, and is thus referred to as the scattering plane.

The term polarization refers to the time-varying behavior of $\mathbf{E}$ at a given point in space, as shown in Fig. 3.2, where the tip of $\mathbf{E}$ over time is a path that lies in the plane perpendicular to the direction of propagation. In general, the components $E_v$ and $E_h$ may be expressed as

$$E_v = a_v \exp[-i(\xi + \delta_v)]$$ (3.2)

$$E_h = a_h \exp[-i(\xi + \delta_h)]$$ (3.3)
Figure 3.1 Propagation of an electromagnetic wave. The inset shows how the electric vector is decomposed into two orthogonal components.

Figure 3.2 Polarization ellipse. $a_h$ and $a_v$ are the amplitudes of the horizontal and vertical components of the electric field, $b$ and $c$ are the lengths of the semi-major and -minor axes, $\chi$ is the orientation angle between the $E_v$ and $E_x$ axes, and $\beta$ is the ellipticity tangent that satisfies $\tan \beta = \pm c/b$, where the sign is positive for right-handed polarization and negative for left-handed polarization.
where $a_v$ and $a_h$ are amplitudes, $\delta_v$ and $\delta_h$ are phases, $\xi = kx - \omega t$, $k (= 2\pi/\lambda)$ is the propagation constant (or wavenumber) at wavelength $\lambda$, and $\omega (= k c_0)$ is the circular frequency with the speed of light in a vacuum of $c_0$. Using these equations, it may be shown (Liou, 1980) that, in general, the path of the tip of the electric field vector is defined by the equation of an ellipse. Some of the parameters used to describe the polarization ellipse in Fig. 3.2 are the orientation angle $\chi$, the ellipticity $\beta$, the semi-major and -minor axes lengths $b$ and $c$, and the electric field amplitudes $a_v$ and $a_h$. The polarization is called right-handed for $\tan \beta > 0$ and left-handed for $\tan \beta < 0$ and corresponds to the helix traced in space by the tip of the electric field vector. This handedness does not depend on the viewing direction, because, a right-handed helix, like the threads of a screw, appear right-handed independent of the observers viewpoint. Finally, notice that two sets of basis vectors are shown for the electric field: $\mathbf{E}_x$ and $\mathbf{E}_y$ are defined by a plane referred to as the meridional plane (defined by the $y$-axis and the direction of travel), while $\mathbf{E}_h$ and $\mathbf{E}_v$ are defined by the scattering plane (defined by the $v$-axis and the direction of travel). This is discussed further when the VRTE is developed.

### 3.2 Stokes Parameters

The rate of flow of radiant energy across a unit area perpendicular to the direction of travel per unit solid angle and per unit wavelength interval is known as the radiant intensity. Because of the transverse nature of electromagnetic waves (time-varying behavior of the electric and magnetic fields), intensity alone is insufficient to completely characterize radiant energy. However, for the complex electric field vector described by vertical and horizontal components $E_v$ and $E_h$ with phase difference $\delta (= \delta_h - \delta_v)$, the radiant energy may be fully described by the Stokes vector

$$\mathbf{T} = (I, Q, U, V)^T$$  \hspace{1cm} (3.4)

The Stokes parameters are defined for the plane wave $\mathbf{E}$ propagating through a differential solid angle $d\Omega$ in a medium with intrinsic impedance $\eta$ and are defined by (Ulaby, et al., 1981)

$$I \, d\Omega = \frac{1}{2\eta} \langle E_v E_v^* + E_h E_h^* \rangle = \frac{1}{2\eta} \langle |E_v|^2 + |E_h|^2 \rangle$$  \hspace{1cm} (3.5)

$$Q \, d\Omega = \frac{1}{2\eta} \langle E_v E_v^* - E_h E_h^* \rangle = \frac{1}{2\eta} \langle |E_v|^2 - |E_h|^2 \rangle$$  \hspace{1cm} (3.6)

$$U \, d\Omega = \frac{1}{2\eta} \langle E_v E_h^* + E_h E_v^* \rangle = \frac{1}{\eta} \langle |E_v||E_h|\cos \delta \rangle$$  \hspace{1cm} (3.7)
\[ V \frac{d\Omega}{2\eta} \langle E_v^* E_h^* - E_h E_v^* \rangle = \frac{1}{\eta} \langle |E_v||E_h| \sin \delta \rangle \] (3.8)

The angular brackets in the Eqs. (3.5) - (3.8) indicate a time average over an interval greater than \(10^{-13}\) s (Bickel and Bailey, 1985), and the superscript * denotes the complex conjugate. The Stokes parameters represent, respectively, the radiant intensity, the amount of polarization, the plane of polarization, and the ellipticity of the energy beam as a function of incoming and outgoing directions, and position. These parameters are useful because they can be measured using optical elements, such as polarizers and quarter wave plates (Bickel and Bailey, 1985).

The reason that time-averaging is used in Eqs. (3.8) - (3.8) is that, in general, electromagnetic waves are never strictly monochromatic, that is, they are usually pseudomonochromatic with a bandwidth \(\Delta \omega\), and thus \(a_v, a_h,\) and \(\delta\) are all slowly varying functions of time. According to Bohren and Huffman (1983), in such a wave, if the electric amplitudes fluctuate in time independently of each other, they are then completely uncorrelated, and the wave is said to be unpolarized. If \(a_v\) and \(a_h\) are partially correlated, however, the wave is partially polarized, and if they are completely correlated, the wave is said to be completely polarized. Coulson (1988) defines unpolarized energy as energy that shows complete symmetry around the direction of propagation, and partial polarization as a mixture of streams of both polarized and unpolarized energy. In any case, for quasi-monochromatic waves, it may be shown (Bohren and Huffman, 1983) that

\[ I^2 \geq Q^2 + U^2 + V^2 \] (3.9)

This condition becomes a strict equality when energy is completely polarized, and if the energy is completely unpolarized, then \(Q = U = V = 0\). This observation leads to the notion of the "degree of polarization" defined as

\[ Z = \sqrt{Q^2 + U^2 + V^2} / I \] (3.10)

For complete polarization \(Z = 1\), for unpolarized energy \(Z = 0\), and in general, \(0 \leq Z \leq 1\). Two other quantities that are defined in terms of the Stokes parameters and are used in remote sensing studies are the linear depolarization ratio, which is the ratio of cross-polarized to co-polarized light

\[ \text{LDR} = \frac{I - Q}{I + Q} \] (3.11)

and the circular depolarization ratio, which is the ratio of same-helicity to opposite-helicity light
$$\text{CDR} = \frac{I + V}{I - V}$$  \hspace{1cm} (3.12)

These quantities are typically used in lidar and radar remote sensing studies (Mishchenko and Hovenier, 1995; Ito et al., 1995) to describe backscattered energy. It can be readily verified that, due to Eq. (3.9), the LDR and CDR must be positive.

The Stokes parameters may also be written in terms of the ellipsometric parameters as

$$I = b^2 + c^2$$  \hspace{1cm} (3.13)

$$Q = I \cos 2\beta \cos 2\chi$$  \hspace{1cm} (3.14)

$$U = I \cos 2\beta \sin 2\chi$$  \hspace{1cm} (3.15)

$$V = I \sin 2\beta$$  \hspace{1cm} (3.16)

The relationship between Eqs (3.13) - (3.16) may be visualized using the Poincaré sphere, shown in Fig. 3.3. On this sphere, the radius is given by I, and the triad \((Q, U, V)\) represent the Cartesian coordinates of a point either on or inside of the sphere. If the point is inside the sphere, partially polarized energy is represented, and if the point lies on the surface of the sphere, completely polarized energy is represented. A point at the origin represents unpolarized energy. The northern and southern hemispheres represent right-handed and left-handed elliptic polarizations, respectively, and the equatorial plane represents linear polarization. The zenith and azimuthal angles are given by \(\pi/2 - 2\beta\) and \(2\chi\), respectively.

Sometimes in the literature an alternate Stokes vector is given as

$$\overline{I}_* \equiv (I_v, I_h, U, V)^T$$  \hspace{1cm} (3.17)

where the subscript '\(*\)' denotes alternate. This set is related to \((I, Q, U, V)^T\) by

$$I_v = I + Q$$  \hspace{1cm} (3.18)

$$I_h = I - Q$$  \hspace{1cm} (3.19)

where \(I_v\) and \(I_h\) are referred to as the vertically and horizontally polarized intensities, and \(U\) and \(V\) remain as previously defined.
Figure 3.3  Polarization representation using a Poincaré sphere. The point shown has rectangular coordinates (U, Q, V), and the sphere has a radius of I. The northern and southern hemispheres represent left-handed and right-handed elliptic polarizations and the equator represents linear polarization. Points inside the sphere correspond to partially polarized light, and unpolarized light is represented by the center of the sphere. The zenith and azimuthal angles are given by $\pi/2 - 2\beta$ and $2\chi$, respectively.

3.3 Mueller Matrix

For an electromagnetic wave with electric field components $(E_{vi}, E_{hi})$ incident on a particle of arbitrary shape and size, the scattered electric field $(E_{vs}, E_{hs})$ at a distance $r$ from the particle is given by the far-field ($kr\gg 1$) expression (Bohren and Huffman, 1983)

$$
\begin{pmatrix}
E_{vs} \\
E_{hs}
\end{pmatrix}
= \frac{\exp[ik(r-y)]}{-ikr}
\begin{pmatrix}
S_2 & S_3 \\
S_4 & S_1
\end{pmatrix}
\begin{pmatrix}
E_{vi} \\
E_{hi}
\end{pmatrix}
$$

(3.20)

where $y$ is the vertical direction in Cartesian coordinates and $S_j$ ($j = 1,2,3,4$) are the complex amplitude functions that depend on the scattering angle and particle size and shape. Equation (3.20) states that the amplitude of the scattered electric field is a linear function of the amplitude of the incident electric field. Computation of $S_j$ is of primary importance because, as is shown later, the radiative properties used in the RTE and VRTE are obtained from these amplitudes. It should be noted that the $S_j$ differ from $S'$ defined by Eq. (3.1).
In terms of the Stokes vector \( \mathbf{T} = (I, Q, U, V)^T \), the relationship between incident and scattered radiation may be expressed mathematically as

\[
\mathbf{I}_s = \frac{1}{k_{2g}^2} \mathbf{M} \mathbf{I}_i
\]

where the scattering transformation matrix \( \mathbf{M} \) is the \( 4 \times 4 \) matrix Mueller matrix, also commonly referred to as the Stokes scattering matrix, whose elements can be written in partitioned matrix form as

\[
\mathbf{M} = \begin{pmatrix}
\overline{A}_{11} & \overline{A}_{12} \\
\overline{A}_{21} & \overline{A}_{22}
\end{pmatrix}
\]

where the submatrices \( \overline{A}_{jk} \) are written in terms of the scattering amplitude functions as (see Appendix A of Coulson, 1988)

\[
\overline{A}_{11} = \frac{1}{2} \begin{pmatrix}
(I|S_1|^2 + |S_2|^2 + |S_3|^2 + |S_4|^2) & (|S_2|^2 - |S_1|^2 + |S_4|^2 - |S_3|^2) \\
(|S_2|^2 - |S_1|^2 - |S_4|^2 + |S_3|^2) & (|S_1|^2 + |S_2|^2 - |S_3|^2 - |S_4|^2)
\end{pmatrix}
\]

\[
\overline{A}_{12} = \begin{pmatrix}
\text{Re}(S_2 S_3^* + S_1 S_4^*) & \text{Im}(S_2 S_3^* - S_1 S_4^*) \\
\text{Re}(S_2 S_3^* - S_1 S_4^*) & \text{Im}(S_2 S_3^* + S_1 S_4^*)
\end{pmatrix}
\]

\[
\overline{A}_{21} = \begin{pmatrix}
\text{Re}(S_2 S_4^* + S_1 S_3^*) & \text{Re}(S_2 S_4^* - S_1 S_3^*) \\
\text{Im}(S_2 S_4^* + S_1 S_3^*) & \text{Im}(S_2 S_4^* - S_1 S_3^*)
\end{pmatrix}
\]

\[
\overline{A}_{22} = \begin{pmatrix}
\text{Re}(S_1 S_2^* + S_3 S_4^*) & \text{Im}(S_1 S_2^* + S_3 S_4^*) \\
\text{Im}(S_1 S_2^* - S_3 S_4^*) & \text{Re}(S_1 S_2^* - S_3 S_4^*)
\end{pmatrix}
\]

where \( \text{Re} \) and \( \text{Im} \) denote the real and imaginary parts, respectively. The elements of the Mueller matrix are determined using single particle scattering theory, which is discussed in Chapter IV. Because there are sixteen elements in the Mueller matrix that are constructed from four amplitudes and three phase differences, there must be nine independent relationships between these elements (van de Hulst, 1981). These relationships are presented explicitly by Fry and Kattawar (1981) and are shown to be equalities for
scattering by a single particle in a fixed orientation, and inequalities for a polydispersion of particles.

If an incident beam passes through a succession of \( n \) particles whose Mueller matrices are given by \( \overline{M}_1, \overline{M}_2, \ldots, \overline{M}_n \), then the effective Mueller matrix is obtained by multiplying together the Mueller matrices associated with each element, that is,

\[
\overline{M} = \overline{M}_n \overline{M}_{n-1} \ldots \overline{M}_1 = \prod_{i=1}^{n} \overline{M}_{n-i+1}
\] (3.24)

As matrix multiplication is not commutative, the matrices must be multiplied as shown.

### 3.4 Vector Radiative Transport Equation

For unpolarized monochromatic radiation propagating along a line of sight distance \( s \), the change in radiative intensity is given by the RTE, stated for three-dimensional Cartesian geometries as

\[
\frac{d}{ds} I(x, y, z; \mu, \phi) = -k_e(x, y, z) I(x, y, z; \mu, \phi) + k_a(x, y, z) I_b(x, y, z)
\]

\[
+ \frac{k_s(x, y, z)}{4\pi} \int_{0}^{2\pi} \int_{-1}^{1} I(x, y, z; \mu', \phi') \Phi(x, y, z; \mu, \phi; \mu', \phi') \, d\mu' \, d\phi' \quad (3.25)
\]

where \( k_a \) and \( k_s \) are the absorption and scattering coefficients, \( k_e (= k_a + k_s) \) is the extinction coefficient, \( I_b \) is the blackbody intensity given by Planck's law (defined in Section 3.6), \( \Phi \) is the scattering phase function, \((\mu, \phi)\) represents the direction of propagation, and the primes denote incoming quantities. The integration in Eq. (3.25) is over the entire sphere surrounding the location under consideration. The terms on the right hand side of Eq. (3.25) represent, respectively, the attenuation of intensity along \( s \) due to absorption and outward scattering, enhancement of intensity through emission due to temperature, and enhancement of intensity due to scattering of incoming radiation into the line of sight direction.

For polarized quasi-monochromatic radiation, an analogous set of equations results by considering that, for a mixture of scatterers, the Stokes parameters of the mixture are the sum of the respective Stokes parameters of the individual scatterers. This is called "incoherent addition of Stokes parameters," and is valid for independent scatterers (van de Hulst, 1981; Tsang et al., 1985). Using this incoherent addition property, the VRTE is written as (Gasiewski, 1993)
\[ \frac{d}{ds} \mathcal{T}(x, y, z; \mu, \phi) = -\overline{k_e}(x, y, z; \mu, \phi) \mathcal{T}(x, y, z; \mu, \phi) + \overline{k_a}(x, y, z; \mu, \phi) I_b(x, y, z) \]

\[ + \int_0^{2\pi} \int_0^1 \overline{Z}(x, y, z; \mu, \phi; \mu', \phi') \mathcal{T}(x, y, z; \mu', \phi') \, d\mu' \, d\phi' \]  

(3.26)

A partial derivation of this equation is given in Appendix C. A complete derivation of Eq. (3.26) is a tedious process, but details for a Rayleigh atmosphere (Appendix D.2.3) are given by Chandrasekhar (1960), for scattering by non-emitting particles by Ishimaru and Cheung (1980), and for the extension to emitting particles by Tsang (1984). A comparison of Eq. (3.26) with Eq. (3.25) shows that the scalar intensity of the RTE has been replaced by the Stokes intensity vector; the position-dependent scalar extinction coefficient of the RTE has been replaced by a direction- and position-dependent extinction matrix; the position-dependent scalar absorption coefficient of the RTE has been replaced by a direction- and position-dependent emission (absorption) vector; the scalar phase function has been replaced by the rotated scattering phase matrix; and the scattering coefficient of the RTE has been absorbed into the phase matrix.

3.5 Scattering Phase Matrix

3.5.1 Definition

For an ensemble of particles, the scattering phase matrix is defined by averaging the Mueller scattering matrix over particle size and orientation according to

\[ \overline{\mathcal{P}}(\mu, \phi; \mu', \phi') = \frac{1}{k^2} \langle N(r, \Xi) \overline{\mathcal{M}}(\mu, \phi; \mu', \phi'; r, \Xi) \rangle \]  

(3.27)

The expression \( N(r, \Xi) \) refers to the particle size distribution, which, in addition to being a function of radius, is also a function of particle orientation \( \Xi \). The operator \( \langle \rangle \) indicates that the quantity is to be averaged over particle size and orientation. For example, for spherical particles, the operator \( \langle \rangle \) means \( \int_0^\rho [\cdot] N(r) \, dr \). The operation performed in Eq. (3.27) ensures that the scattering phase matrix has appropriate units for use in the VRTE. However, the matrix must first be rotated as explained next.

24
3.5.2 Rotation of Reference Frame

The scattering phase matrix is defined in terms of the scattering plane while the VRTE defined in Section 3.4 is defined in terms of its own Cartesian reference frame. In this section, a relationship between the scattering phase matrix and the rotated scattering phase matrix needed by the VRTE is developed.

Consider Fig. 3.4, where the geometry for the polarization reference frame is shown. In this figure, the meridian plane P1OY is the plane of the incoming wave $\overline{T}_s$ that generates the scattering source term. The scattering phase matrix is defined in terms of the scattering plane P1OP2, and the outgoing intensity $\overline{T}_i$ is defined in terms of the meridian plane P2OY. Therefore, for the VRTE, the polarization must be rotated into the scattering plane P1OP2 by angle $i_1$, the scattering phase matrix is then applied, and finally the polarization is rotated out of the plane by angle $i_2$ to obtain the rotated scattering phase matrix. In mathematical terms, this transformation is expressed by

$$\overline{Z}(\mu, \phi; \mu', \phi') = \overline{L}(\pi - i_2) \overline{P}(\mu, \phi; \mu', \phi') \overline{L}(-i_1)$$

(3.28)

where the (I, Q, U, V) Stokes basis polarization rotation matrix is (Liou, 1980)

$$\overline{L}(\zeta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\zeta & \sin 2\zeta & 0 \\
0 & -\sin 2\zeta & \cos 2\zeta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(3.29)

Using spherical trigonometry (see Appendix F of Liou, 1980), $i_1$ and $i_2$ are obtained as

$$\cos i_1 = \frac{-\mu + \mu' \cos \Theta}{\pm (1 - \cos^2 \Theta)^{1/2} (1 - \mu^2)^{1/2}}$$

(3.30a)

$$\cos i_2 = \frac{-\mu' + \mu \cos \Theta}{\pm (1 - \cos^2 \Theta)^{1/2} (1 - \mu^2)^{1/2}}$$

(3.30b)

where $\Theta$ is the scattering angle between incoming and outgoing beams and is given by

$$\cos \Theta = \mu \mu' + (1 - \mu^2)^{1/2} (1 - \mu'^2)^{1/2} \cos(\phi' - \phi)$$

(3.30c)
Figure 3.4 Geometry for polarization reference frame. For a scatterer at the origin, the scattering plane is OP
1P
2, the meridian plane for the incident ray (θ', φ') is P
1OY, and that for the outgoing ray (θ, φ) is P
2OY.

In the denominator of Eqs. (3.30a) and (3.30b), the plus sign is taken for π < φ' − φ < 2π and the minus sign is taken for 0 < φ' − φ < π. Additional, though equivalent, expressions formulated by Evans and Stephens (1990) are

\[
\sin i_1 = \frac{(1 - \mu^2)^{1/2} \sin(\phi' - \phi)}{(1 - \cos^2 \Theta)^{1/2}} \tag{3.31a}
\]

\[
\sin i_2 = \frac{(1 - \mu'^2)^{1/2} \sin(\phi' - \phi)}{(1 - \cos^2 \Theta)^{1/2}} \tag{3.31b}
\]

\[
\cos i_1 = \left[\mu (1 - \mu^2)^{1/2} - \mu' (1 - \mu'^2)^{1/2} \cos(\phi' - \phi)\right] / (1 - \cos^2 \Theta)^{1/2} \tag{3.31c}
\]

\[
\cos i_2 = \left[\mu' (1 - \mu'^2)^{1/2} - \mu (1 - \mu^2)^{1/2} \cos(\phi' - \phi)\right] / (1 - \cos^2 \Theta)^{1/2} \tag{3.31d}
\]

3.5.3 Normalization

The purpose of this section it to show the relationship between the Mueller scattering matrix and the scattering phase function used in the scalar RTE. The normalized
Mueller matrix $\Phi$ may be introduced by requiring that its first element satisfy the normalization requirement (Liou, 1992)

$$\int_{4\pi} \Phi_{11} \, d\Omega = 4\pi$$  \hspace{1cm} (3.32)

The element $\Phi_{11}$ appearing in Eq. (3.32) is equivalent to the phase function $\Phi$ of Eq. (3.25). Defining the scattering cross section as

$$C_s = \frac{1}{k^2} \int_{4\pi} M_{11} \, d\Omega$$  \hspace{1cm} (3.33)

the normalized Mueller matrix can be defined in terms of the Mueller matrix as

$$\frac{\Phi}{4\pi} = \frac{M}{C_s \, k^2}$$  \hspace{1cm} (3.34)

The scattering cross section $C_s$ is related to the scattering coefficient $k_s$ used in Eq. (3.32) and, for spherical particles, the relationship is given in Chapter IV. Inserting Eq. (3.34) into Eq. (3.21) allows the relationship between incident and scattered Stokes vectors to be rewritten in terms of the normalized Mueller matrix as

$$\overline{I_s} = d\Omega \frac{\Phi}{4\pi} \overline{I_i}$$  \hspace{1cm} (3.35)

where $d\Omega = C_s / r^2$ is the differential solid angle associated with the scattering.

### 3.6 Other Stokes Bases

For other Stokes bases, for example, the ($I_V$, $I_h$, $U$, $V$) basis, the vectors and matrices used in the VRTE differ from those used in this study (e.g. see Chandrasekhar, 1960). Expressions for transformations between the Stokes bases $\overline{I_s}$ and $\overline{I}$ are given by van de Hulst (1980) and are reproduced here for convenience. Any results expressed in one of these sets can be transformed into the same results in terms of the other set using

$$\overline{I} = \overline{W} \overline{I_s}, \quad \overline{I_s} = \overline{W}^{-1} \overline{I}$$  \hspace{1cm} (3.36)

where the basis transformation matrices are
\[
\mathbf{\overline{W}} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad \mathbf{\overline{W}}^{-1} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\] (3.37)

Also, a matrix multiplication of the form \( \mathbf{\overline{K}} = \mathbf{\overline{Z}}^{-1} \mathbf{\overline{T}} \) is equivalent to \( \mathbf{\overline{K}}_* = \mathbf{\overline{Z}}_* \mathbf{\overline{T}}_* \) if the preceding transformation is applied to the vectors \( \mathbf{\overline{T}} \) and \( \mathbf{\overline{K}} \) and the matrix is transformed via

\[
\mathbf{\overline{Z}} = \mathbf{\overline{W}} \mathbf{\overline{Z}}_* \mathbf{\overline{W}}^{-1}, \quad \mathbf{\overline{Z}}_* = \mathbf{\overline{W}}^{-1} \mathbf{\overline{Z}} \mathbf{\overline{W}}
\] (3.38)

An example is given in Appendix D.

### 3.7 Brightness Temperature

A blackbody is an idealized body that absorbs all energy that is incident upon it, and emits energy according to the Planck law

\[
E_B = \frac{2 \pi C_1}{\lambda^5 \left[ \exp \left\{ \frac{C_2}{(\lambda \ T)} \right\} - 1 \right]} \] (3.39)

where \( E_B \) is the spectral hemispherical blackbody emissive power, \( C_1 = 0.59544 \times 10^8 \ \text{W} \cdot \mu\text{m}^4/\text{m}^2 \), \( C_2 = 1.4388 \times 10^4 \ \mu\text{m} \cdot \text{K} \), \( T \) is the local temperature in degrees K, and \( \lambda \) is the wavelength in units of \( \mu\text{m} \). Because blackbody radiation is isotropic (that is, the blackbody intensity is independent of direction and unpolarized), the spectral blackbody intensity is related to the spectral hemispherical blackbody emissive power by

\[
I_B = \frac{E_B}{\pi} \] (3.40)

For a given wavelength and temperature, real bodies emit less energy than a blackbody. The temperature at which a blackbody would emit an equivalent amount of energy as that for a real body is known as the brightness temperature. Mathematically, this relationship is expressed by inverting Eq. (3.39) to solve for the temperature as a function of fixed wavelength and direction dependent intensity, so that

\[
T_B(\mu, \phi) = \frac{C_2}{\lambda \ln \left[ \frac{1}{1 + 2 C_1 / \left( \lambda^5 I(\mu, \phi) \right)} \right]} \] (3.41)
where $I(\mu, \phi)$ is the intensity from the real body. For long wavelengths and/or high temperatures, $C_2 / (\lambda T) \ll 1$, and the Rayleigh-Jeans approximation for the brightness temperature applies, and is written as

$$T_{B,RJ}(\mu, \phi) = \frac{C_2}{2C_1} \lambda^4 I(\mu, \phi)$$

This approximation is often employed in the microwave, where, for example, it is accurate to within about 2 K at 85 GHz (Haferman, et al., 1993c).

### 3.8 Radiant Flux

In many applications, it is desirable to consider radiant energy in the examination of energy conservation within a medium. For example, in atmospheric applications, one might desire to consider the net solar radiant energy supplied to a cloud. In heat transfer engineering applications, an energy balance incorporating radiant energy components may be needed to obtain temperature distributions within a medium. A detailed discussion of radiant energy conservation with derivations is given by Siegel and Howell (1992), who define the net radiant energy leaving a unit volume as the divergence of radiant flux vector, given for 3-D Cartesian geometries by

$$\nabla \cdot \bar{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

where the components of the net radiant flux vector are

$$q_n = \int_0^{4\pi} I \nu \, d\Omega$$

In Eq. (3.44), $n$ is one of $x$, $y$, or $z$, $\nu$ is the cosine of the angle between the direction of $I$ and the normal of the $x$-, $y$-, or $z$-face under consideration, and $I$ is the total intensity, defined as

$$I = \int_0^\infty I_{\lambda} \, d\lambda$$

where $I_{\lambda}$ is the spectral intensity at wavelength $\lambda$ ($I_{\lambda}$ is the intensity that appears in the RTE and VRTE of Eqs. (3.25) and (3.26), but for brevity, the subscript $\lambda$ does not appear in those equations). Similarly, spectral net fluxes may be written as
\[ q_{\lambda n} = \int_0^{4\pi} I_\lambda \, n \, d\Omega \quad (3.46) \]

The total and spectral fluxes may also be broken up into hemispherical or partial fluxes by changing the limits on the directional integration. For example, the spectral flux into the forward hemisphere of the x-face is

\[ q_{\lambda x}^+ = \int_0^{2\pi} \int_0^1 I_\lambda \, \mu \, d\mu \, d\phi \quad (3.47) \]

Using Eqs. (3.44) and (3.45), Eq. (3.25) is substituted into Eq. (3.43) to yield

\[
\nabla \cdot \bar{q} = \int_0^\infty \left[ -k_e \int_0^{4\pi} I_\lambda(\Omega') \, d\Omega + k_a \int_0^{4\pi} I_b \, d\Omega \right. \\
\left. + \frac{k_s}{4\pi} \int_0^{4\pi} \int_0^{4\pi} I_\lambda(\Omega') \Phi(\Omega', \Omega) \, d\Omega' \, d\Omega \right] d\lambda \quad (3.48)
\]

The preceding equation is the general expression for the divergence of radiant flux for unpolarized radiation. The integration of \( I_b \) over \( \Omega \) is readily performed, but is retained for display purposes. If scattering is independent of incident direction (for example, for randomly oriented axisymmetric particles), then Eq. (3.48) reduces to

\[
\nabla \cdot \bar{q} = \int_0^\infty \left[ -k_a \int_0^{4\pi} I_\lambda(\Omega') \, d\Omega + k_a \int_0^{4\pi} I_b \, d\Omega \right] d\lambda \quad (3.49)
\]

To properly account for polarized radiation, Eq. (3.26) is substituted into Eq. (3.43) so that the divergence of the radiant flux for the first Stokes parameter is written as

\[
\nabla \cdot \bar{q}_1 = \int_0^\infty \left[ -\int_0^{4\pi} (k_{e11} I_\lambda + k_{e12} Q_\lambda + k_{e13} U_\lambda + k_{e14} V_\lambda) \, d\Omega + \int_0^{4\pi} k_{a1} I_b \, d\Omega \right. \\
\left. + \frac{1}{4\pi} \int_0^{4\pi} \int_0^{4\pi} (Z_{11} I_\lambda + Z_{12} Q_\lambda + Z_{13} U_\lambda + Z_{14} V_\lambda) \, d\Omega' \, d\Omega \right] d\lambda \quad (3.50)
\]
Divergence of radiant flux expressions may be written for the remaining Stokes parameters. Equation (3.50) is perhaps the most important because the first Stokes parameter is the magnitude of radiant intensity, and thus, Eq. (3.50) is appropriate for use in energy balances. Note that Eq. (3.50) simplifies to Eq. (3.49) for the special case of unpolarized radiation.
CHAPTER IV

RADIATIVE PROPERTIES

This chapter discusses the radiative properties that are needed for the equations of transfer (both scalar and vector). As discussed by Ulaby et al. (1981, p. 1091), the radiative transfer method discussed in Chapter III is valid when the spacing between scatterers is large. They cite studies that show the spacing between scatterers must be larger than $\lambda/3$ and $0.8r$, where $\lambda$ is the wavelength under consideration and $r$ is the radius of the scatterer. Under these conditions, the effective radiative coefficients for a differential volume are obtained by summing the radiative coefficients of the individual species. For example, if a volume consists of atmospheric gases, clouds, and precipitation, then the generic radiative coefficient $\gamma$ is given by

$$\gamma = \gamma_g + \gamma_c + \gamma_p$$

(4.1)

where the subscripts $g$, $c$, and $p$ refer to gases, clouds, and precipitation, respectively, and $\gamma$ is one of $k_a$, $k_e$, or $k_s$ of Eq. (3.25).

To determine radiative coefficients, the complex refractive index $m = m_r + i m_i$ of the constituent of interest must be known. The real part $m_r$ of the complex refractive index is related to how fast light propagates through the particular species, and the imaginary part $m_i$ is related to how much energy the species absorbs. It should be emphasized that, for a specified material, the complex refractive index is a function of temperature and wavelength. Once the complex refractive index is known, electromagnetic theory may be used to determine the radiative properties of a particular species, based on its shape and size parameter, $X$, defined as the ratio of a characteristic circumference of a particle to incident wavelength. For a spherical particle of radius $r$ and wavelength $\lambda$, $X = 2 \pi r / \lambda$ (= $k r$, where the wavenumber $k$ has been defined in Section 3.1).

The following sections describe how radiative properties of various atmospheric particles and gases may be determined, and also provide mathematical descriptions of the radiative properties of surfaces. It should be noted that entire books have been written on these subjects (for example, van de Hulst, 1981; Ulaby et al., 1981; Bohren and Huffman, 1983; Tsang et al., 1985), and therefore, for more details, the interested reader should consult an appropriate reference.
4.1 Spherical Particles

In general, when an electromagnetic wave interacts with a particle, the energy is redistributed through scattering and absorption. The ratio of the rate at which energy is absorbed to the incident energy per unit area is known as the absorption cross section $C_a$, which has units of area. Similarly, the scattering cross section $C_s$ may be defined, and then, the extinction cross section is defined as $C_e = C_a + C_s$. Efficiency factors are defined as the ratio of the cross section to the geometric cross section, and, for spherical particles of radius $r$, are given by

$$Q_i = C_i / (\pi r^2)$$  \hspace{1cm} (4.2)

where the subscript $i$ denotes one of a, e, or s. It is emphasized that the efficiency factors are functions of temperature, wavelength, composition, and radius. In functional form, this may be written as $Q_i = Q_i(X, m)$.

To obtain expressions for the efficiency factors, it is necessary to solve Maxwell's equations. An exact solution to Maxwell's equations for the problem of scattering and absorption of electromagnetic waves by a spherical particle is provided by the Mie theory (Bohren and Huffman, 1983). According to the Mie theory, the scattering efficiency factor is evaluated as

$$Q_s = \frac{2}{X^2} \sum_{n=1}^{\infty} (2n + 1) \left[ |a_n|^2 + |b_n|^2 \right]$$  \hspace{1cm} (4.3)

where the coefficients $a_n$ and $b_n$ are determined from boundary condition requirements and are normally written in terms of Bessel and Hankel functions (for example, see Liou, 1992). However, for computational purposes, Bohren and Huffman (1983) give these coefficients as

$$a_n = \frac{\left( \frac{d_n}{m} + \frac{n}{X} \right) \Psi_n(X) - \Psi_{n-1}(X)}{\left( \frac{d_n}{m} + \frac{n}{X} \right) \Pi_n(X) - \Pi_{n-1}(X)}$$  \hspace{1cm} (4.4a)

$$b_n = \frac{\left( \frac{d_n}{m} + \frac{n}{X} \right) \Psi_n(X) - \Psi_{n-1}(X)}{\left( \frac{d_n}{m} + \frac{n}{X} \right) \Pi_n(X) - \Pi_{n-1}(X)}$$  \hspace{1cm} (4.4b)

where $n = 1$ to $N$, with $N = \max(X + 4X^{1/3} + 2, |Y|)$ and $Y = mX$. The backward recurrence relation for $d_n$ is given by
\[ d_{n-1} = \frac{n}{Y} - \frac{1}{d_n + n/Y} \]  

(4.5)

where the recurrence is started with \( d_N = 0 \). The other quantities in Eq. (4.4) are defined by

\[ \Psi_n(X) = \frac{2n + 1}{X} \Psi_{n-1}(X) - \Psi_{n-2}(X) \]  

(4.6a)

\[ \Pi_n(X) = \Psi_n(X) - i \gamma_n(X) \]  

(4.6b)

where

\[ \gamma_n(X) = \frac{2n + 1}{X} \gamma_{n-1}(X) - \gamma_{n-2}(X) \]  

(4.7)

and

\[ \Psi_{-1}(X) = \cos X \quad \Psi_0(X) = \sin X \]
\[ \gamma_{-1}(X) = -\sin X \quad \gamma_0(X) = \cos X \]  

(4.8)

The scattering amplitude functions introduced in Eq. (3.20) are given by

\[ S_1(\Theta) = \sum_{n=1}^{N} \frac{2n + 1}{n(n+1)} \left[ a_n \pi_n(\Theta) + b_n \tau_n(\Theta) \right] \]  

(4.9a)

\[ S_2(\Theta) = \sum_{n=1}^{N} \frac{2n + 1}{n(n+1)} \left[ b_n \pi_n(\Theta) + a_n \tau_n(\Theta) \right] \]  

(4.9b)

and \( S_3 = S_4 = 0 \). In Eq. (4.9), the scattering angle-dependent coefficients \( \pi_n \) and \( \tau_n \) are computed from the recurrence relations

\[ \pi_n(\Theta) = \frac{2n - 1}{n - 1} \cos \Theta \pi_{n-1}(\Theta) - \frac{n}{n - 1} \cos \Theta \pi_{n-2}(\Theta) \]  

(4.10a)

\[ \tau_n(\Theta) = n \cos \Theta \pi_n(\Theta) - (n + 1) \pi_{n-1}(\Theta) \]  

(4.10b)

where the recurrence is begun with \( \pi_0 = \tau_0 = 0 \) and \( \pi_1 = \tau_1 = 1 \). The extinction efficiency is evaluated from

\[ Q_e = \frac{4}{X^2} \text{Re}[S(0)] = \frac{2}{X^2} \sum_{n=1}^{\infty} (2n + 1) \text{Re}(a_n + b_n) \]  

(4.11)
where \( S(0) = S_1(0) = S_2(0) \) and \( \text{Re} \) denotes the real part. Finally, the phase function is determined from

\[
\Phi(\Theta, r) = \frac{4}{Q_s X^2} S_{11}(\Theta)
\]  

(4.12)

where

\[
S_{11}(\Theta) = \frac{1}{2} \left[ |S_1(\Theta)|^2 + |S_2(\Theta)|^2 \right]
\]  

(4.13)

The preceding discussion applies to a single spherical particle and wavelength. For a polydispersion of independent spherical particles, the generalized spectral radiative coefficients are computed using (Liou, 1992)

\[
\gamma_i = \int_{r_{\min}}^{r_{\max}} C_i(r) N(r) \, dr = \int_{r_{\min}}^{r_{\max}} Q_i(r) \pi r^2 N(r) \, dr
\]  

(4.14)

where \( N(r) \) is the particle number density (i.e. DSD) that gives the number of particles per unit volume with radii between \( r \) and \( r + dr \), and has units of inverse volume times inverse length. In Eq. (4.14), \( r_{\min} \) and \( r_{\max} \) are the lower and upper limit of the particle radius. The phase function for a polydispersion is computed from

\[
\Phi(\Theta) = \frac{1}{k_s} \int_{r_{\min}}^{r_{\max}} C_s(r) \Phi(\Theta, r) N(r) \, dr = \frac{1}{k_s} \int_{r_{\min}}^{r_{\max}} Q_s(r) \Phi(\Theta, r) \pi r^2 N(r) \, dr
\]  

(4.15)

Other quantities of interest include the optical depth, mentioned in Chapter I, and defined by

\[
\tau = \int_{s_1}^{s_2} k_e(s) \, ds
\]  

(4.16)

where \( s \) is a spatial coordinate, the single scattering albedo, given by

\[
\omega = \frac{k_s}{k_e} = \frac{k_s}{k_a + k_s}
\]  

(4.17)

and the asymmetry parameter that represents the relative strength of forward scattering, and is defined as the first moment of the phase function, that is
\[ g = \frac{1}{2} \int_{-1}^{1} \Phi(\Theta) \cos \Theta \, d(\cos \Theta) \]  

(4.18)

To conclude this section, the results of single scattering computations for spherical water and ice droplets at various microwave frequencies are shown in Figs. 4.1 - 4.4. The properties are computed as prescribed above, where the DSD is written as a function of rainrate through the Marshall-Palmer distribution (Wilheit et al., 1977)

\[ N(r) = N_0 \exp (-\Lambda \, r) \]  

(4.19)

where \( N_0 = 0.16 \text{ cm}^{-4} \), \( \Lambda \) is the exponential parameter, and \( r \) is the drop radius in centimeters. The exponential parameter is given in terms of the rainrate as

\[ \Lambda = 81.56 \, R^{-0.21} \]  

(4.20)

where \( R \) has units of mm/h. The frequencies used in Figs. 4.1 - 4.4 are those of the SSM/I. These figures show how the absorption and scattering coefficients, as well as the single scattering albedo and asymmetry parameter, all increase with frequency, for both ice and liquid water. Spencer et al. (1989) considers the theoretical implications of the values of these properties at microwave frequencies.

### 4.2 Atmospheric Gases

Electromagnetic interaction with gaseous molecules consists of absorption and emission, and, typically, is described using quantum mechanics arguments (Ulaby et al., 1981). For example, radiation is absorbed (or emitted) when a molecule makes a transition from a lower (or higher) energy state to a higher (or lower) energy state, where these energy states are described by discrete quantum numbers. Because these transitions occur at sharply defined frequencies in the electromagnetic spectrum, the term "spectral line" is often used to refer to these transition points that are characteristic of a single isolated molecule.

For a gas containing many molecules, various interactions, such as collisions, cause the gas emission and absorption to spread its energy level transitions over a small range of frequencies, rather than at an isolated frequency. This phenomenon is known as "line broadening," and leads to narrow band models. A more detailed discussion of these issues is outside of the scope of this section, but the general expression for the absorption coefficient of a gas at a frequency \( f \) is (Ulaby et al., 1981)
Figure 4.1 Scattering coefficients of Marshall-Palmer distribution of raindrops.

Figure 4.2 Absorption coefficients of Marshall-Palmer distribution of raindrops.
Figure 4.3 Single-scatter albedo for Marshall-Palmer distribution of raindrops.

Figure 4.4 Asymmetry factor for Marshall-Palmer distribution of raindrops.
\[ k_{ag} = \frac{4\pi f}{c_0} \sum_{Im} F(f, f_{Im}) \]  

(4.21)

where \( c_0 \) is the velocity of light in a vacuum, \( \Sigma_{Im} \) is the line-strength, \( F \) is the line-shape function, subscripts \( l \) and \( m \) refer to low and high energy states, respectively, and \( f_{Im} \) is the molecular resonance frequency given by

\[ f_{Im} = \frac{E_m - E_l}{\hbar} \]  

(4.22)

where \( E \) refers to the internal energy of a gas molecule and \( h \) is Planck's constant. In Eq. (4.21), \( \Sigma_{Im} \) is a function of gas density, temperature, and pressure, as well as molecular parameters associated with the energy level transition. Further discussion, references and formulas are given by Ulaby et al. (1981). Formulas for the main microwave gaseous absorbers (water vapor and oxygen) are summarized by Haferman et al. (1993b), and are reproduced in Appendix A.

### 4.3 Extinction Matrix and Emission Vector

In general, in addition to being polydisperse, particles will generally be aspherical and asymmetric, and may have a preferred direction of orientation. For this general case, the extinction matrix for use in Eq. (3.26) is given by the following expression (Tsang et al., 1985; Evans and Vivekanandan, 1990; Mishchenko, 1990a; Evans and Stephens, 1993; Gasiewski, 1993)

\[
\begin{pmatrix}
-\text{Re}(M_{vv}+M_{hh}) & -\text{Re}(M_{vv}-M_{hh}) & -\text{Re}(M_{vh}+M_{hv}) & -\text{Im}(M_{vh}-M_{hv}) \\
-\text{Re}(M_{vv}-M_{hh}) & -\text{Re}(M_{vv}+M_{hh}) & -\text{Re}(M_{vh}-M_{hv}) & -\text{Im}(M_{vh}+M_{hv}) \\
-\text{Re}(M_{hv}+M_{vh}) & -\text{Re}(M_{hv}-M_{vh}) & -\text{Re}(M_{vv}+M_{hh}) & -\text{Im}(M_{vh}-M_{hv}) \\
-\text{Im}(M_{hv}-M_{vh}) & \text{Im}(M_{vh}+M_{hv}) & -\text{Im}(M_{vv}-M_{hh}) & -\text{Re}(M_{vv}+M_{hh})
\end{pmatrix}
\]

\[ + k_{ag} \overline{\delta} \]  

(4.23)

where \( \overline{\delta} \) is the identity matrix. Equation (4.23) states that extinction is due to absorption and scattering due to particles plus absorption due to gases. The symbol \( M_{\alpha\beta} \) used in Eq. (4.23) is defined by

\[ M_{\alpha\beta} = \frac{2\pi}{k^2} \langle N(\tau, \Xi) S_{\alpha\beta} \rangle \]  

(4.24)
where \( S_{\alpha\beta} \) refers to the scattering amplitude functions defined in Chapter III, and the notation \( \alpha\beta \) is defined according to \( S_{vv} \equiv S_2 \), \( S_{vh} \equiv S_3 \), \( S_{hv} \equiv S_4 \), and \( S_{hh} \equiv S_1 \). The operator \( \leftrightarrow \) is defined in Section 3.5.1.

Using the extinction matrix and phase matrix, the emission (absorption) vector is computed as (Evans and Stephens, 1993)

\[
\mathbf{k}_a = \begin{pmatrix}
    \mathbf{k}_{e11} - \int_{4\pi} P_{11} \, d\Omega' \\
    \mathbf{k}_{e12} - \int_{4\pi} P_{12} \, d\Omega' \\
    \mathbf{k}_{e13} - \int_{4\pi} P_{13} \, d\Omega' \\
    \mathbf{k}_{e14} - \int_{4\pi} P_{14} \, d\Omega'
\end{pmatrix}
\]  

(4.25)

Additional details are given in Appendix C.

### 4.4 Boundary Conditions and Surface Properties

The \( \text{VRTE} \) of Eq. (3.26) requires boundary conditions. In this section, expressions describing the effect of emission and reflection from a boundary surface are given (also see Weng, 1992a). In general, the boundary conditions for the \( \text{VRTE} \) are written according to Sánchez et al. (1992) but modified to account for polarization as

\[
\mathbf{T}^+ = \mathbf{e}_0 \mathbf{l}_b + \mathbf{T} \left[ \mathbf{T}^0_{\text{ba}} + \mathbf{T}^0_{\text{c}} \delta(\theta - \theta') \delta(\phi - \phi') \right] + (1 - f_d) \mathbf{R} \mathbf{T}^0_{\text{m}} \n
+ \frac{f_d}{\pi} \left[ \mathbf{R} \mathbf{T}^0_{\text{c}} + \oint \mathbf{v} \mathbf{R} \mathbf{T}^0 \, d\Omega' \right]
\]  

(4.26)

where superscripts +, −, and 0 represent radiation moving from the boundary toward the domain, from the domain toward the boundary, and from outside the boundary toward the domain, respectively. The vectors \( \mathbf{T}^0_{\text{ba}} \) and \( \mathbf{T}^0_{\text{c}} \) denote background and collimated radiation, respectively, \( \mathbf{T}^0_{\text{m}} \) is the Stokes vector in the direction mirroring that of \( \mathbf{T}^+ \), \( f_d \) represents the fraction of radiation that is diffusely reflected, and \( \delta() \) is the Dirac delta function. The symbols \( \mathbf{e}_0 \), \( \mathbf{T} \), and \( \mathbf{R} \) are the emission vector, transmission matrix, and bidirectional reflectance matrix, respectively (for atmospheric applications involving collimated and background radiation at the top of the atmosphere, the transmission matrix is typically taken as the identity matrix). Finally, the symbol \( \mathbf{v} \) is the cosine of the angle between the propagation direction and the normal to the boundary under consideration. Equation (4.26)
states that the Stokes vector for energy leaving a given boundary is due to contributions from energy emitted by the boundary, energy transmitted from outside the boundary due to background and collimated radiation, and specularly and diffusely reflected energy.

4.4.1 Lambertian (Diffuse) Surfaces

A Lambertian, or diffuse, surface is an idealized surface that is "perfectly" rough, and therefore emits and reflects equally in all directions. For such a surface, in which the reflected radiation is assumed unpolarized, the bidirectional reflection matrix is given by

\[
\bar{R} = \begin{pmatrix}
\rho_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (4.27)

where \(\rho_0\) is the hemispherical reflectance at the boundary. The energy emitted by a Lambertian surface is also unpolarized, and therefore, the emission vector is given by

\[
\bar{\varepsilon}_0 = \begin{pmatrix}
\varepsilon_0 \\
0 \\
0 \\
0
\end{pmatrix}
\] (4.28)

where \(\varepsilon_0\) is the unpolarized hemispherical emittance of the boundary. Finally, the transmission matrix is given by

\[
\bar{T} = \begin{pmatrix}
t_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (4.29)

where the unpolarized hemispherical boundary transmittance satisfies \(t_0 = 1 - \varepsilon_0 - \rho_0\).

4.4.2 Fresnel (Specular) Surfaces

A Fresnel, or specular, surface is an idealized surface that is "perfectly" smooth. For such a surface, the reflected radiation is polarized and angle dependent, and the bidirectional reflection matrix is given by (Tsang et al., 1985)
\[ \bar{R} = \begin{pmatrix}
  \frac{1}{2} (|R_v|^2 + |R_h|^2) & \frac{1}{2} (|R_v|^2 - |R_h|^2) & 0 & 0 \\
 0 & 0 & \text{Re}(R_v R_h^*) & -\text{Im}(R_v R_h^*) \\
 \frac{1}{2} (|R_v|^2 - |R_h|^2) & \frac{1}{2} (|R_v|^2 + |R_h|^2) & 0 & 0 \\
 0 & 0 & \text{Im}(R_v R_h^*) & \text{Re}(R_v R_h^*)
\end{pmatrix} \] (4.30)

where the vertical and horizontal reflection coefficients are given by the Fresnel relations

\[ R_v(\theta) = \frac{m^2 \cos \theta - \sqrt{m^2 + \cos^2 \theta - 1}}{m^2 \cos \theta + \sqrt{m^2 + \cos^2 \theta - 1}} \] (4.31a)

\[ R_h(\theta) = \frac{\cos \theta - \sqrt{m^2 + \cos^2 \theta - 1}}{\cos \theta + \sqrt{m^2 + \cos^2 \theta - 1}} \] (4.31b)

and \( m \) is the complex refractive index of the surface. The energy emitted by a Fresnel surface is also polarized and angle dependent, and the emission vector is given by (Evans and Stephens, 1990)

\[ \bar{\varepsilon}^o = \begin{pmatrix}
  1 - \frac{1}{2} (|R_v|^2 + |R_h|^2) \\
  -\frac{1}{2} (|R_v|^2 - |R_h|^2) \\
  0 \\
  0
\end{pmatrix} \] (4.32)

Finally, the transmission matrix for a Fresnel surface is (Tsang et al., 1985)

\[ \bar{T}_* = \begin{pmatrix}
  |R_v|^2 & 0 & 0 & 0 \\
 0 & |R_h|^2 & 0 & 0 \\
 0 & 0 & g_1 & -g_2 \\
 0 & 0 & g_2 & g_1
\end{pmatrix} \] (4.33)

where the subscript * denotes that this matrix is written for the modified Stokes vector, and
\[ g_1 = \frac{\cos \theta_i}{\cos \theta_i} \text{Re}[(1 + R_v)(1 + R_h^*:\ldots)] \tag{4.34a} \]

\[ g_2 = \frac{\cos \theta_i}{\cos \theta_i} \text{Im}[(1 + R_v)(1 + R_h^*)] \tag{4.34b} \]

where \( \theta_i \) is the angle of incidence and \( \theta_t \) is the transmission angle. These two angles are related by Snell's law of refraction,

\[ \sin \theta_i = \frac{m_{r,t}}{m_{r,i}} \sin \theta_t \tag{4.34b} \]

where \( m_{r,t} \) and \( m_{r,i} \) are the real part of the refractive index of the medium from which the wave is traveling to and from, respectively, and \( m_{r,t} > m_{r,i} \). When \( \theta \) is larger than the critical angle given by \( \theta_c = \sin^{-1}[m_{r,t} / m_{r,i}] \), then \( g_1 = g_2 = 0 \).

### 4.4.3 Rough Surfaces

Tsang et al., (pp. 70, 203; 1985) use the theory of Beckmann and Spizzichino (1963) to develop expressions for the reflection and transmission matrices for "random" rough surfaces. Their derivation uses a combination of the Kirchoff approximation and geometrical optics to obtain expressions that include the effects of "shadowing."However, their solution only accounts for single scattering, and, therefore, the reflected and transmitted intensities are always underestimated, unless the surface is only "slightly" rough. Nevertheless, they give a method for determining the upper and lower limits of desired radiant energy quantities emitted by a rough surface. The resulting expressions are quite complicated, and are not reproduced here. For further details, the interested reader should refer to Tsang, et al. (1985).
CHAPTER V
NUMERICAL MODEL AND IMPLEMENTATION

In this chapter, the discrete ordinates method (DOM) used to solve the scalar and vector forms of the RTE is discussed. An overview of the DOM for 1-, 2-, or 3-D geometries under a variety of boundary conditions is presented in Section 5.1, along with the formulation of the vector-DOM (VDOM). Section 5.2 discusses some general computational issues associated with the DOM, and in Section 5.3, the technique used for solving the VDOM using a distributed computing paradigm is described.

5.1 Discrete Ordinates Method

This section describes the DOM. Section 5.1.1 describes how the method is implemented for unpolarized radiative transfer, based on the work of Sánchez et al. (1991), and Section 5.1.2 mentions some special considerations, such as stability and accuracy. Section 5.1.3 derives the VDOM formulation to include polarized radiative transfer.

5.1.1 Background and Formulation

The DOM is a differential approximation to the RTE and is based on the separation of the spatial dependency of the intensity field from the angular dependency. This is accomplished through the selection of multiple discrete directions that satisfy conservation of specific moments of intensity, and certain symmetry conditions. The method was originally developed for application to astrophysical problems (Chandrasekhar, 1960), and soon afterwards was applied to neutron transport problems (Lathrop and Carlson, 1965). In recent years, the method has been used in atmospheric applications (Lenoble, 1985; Stamnes et al., 1988). The DOM has also been applied to the VRTE. For example, Huang and Liou (1983) used the DOM to solve the VRTE for a plane-parallel vertically homogeneous atmosphere consisting of spherical particles. More recently, Weng (1992a,b) used the DOM to solve the VRTE for a plane-parallel vertically inhomogeneous atmosphere by first expanding the radiative intensity in a Fourier cosine series and then replacing the scattering integrals by a Gaussian quadrature.

For the scalar DOM, a set of discrete directions is chosen such that, in three-dimensional Cartesian geometries, the RTE is replaced by the set of partial differential equations
\[ \mu_i \frac{\partial I_i}{\partial x} + \delta_i \frac{\partial I_i}{\partial y} + \gamma_i \frac{\partial I_i}{\partial z} = -k_e I_i + k_a I_b + \frac{k_s}{4\pi} \sum_{j=1}^{8K} w_j I_j \Phi_{ij} + I_c \Phi_{ic} \]  

(5.1)

where \( I_i \) is the full outgoing intensity at position \((x,y,z)\) in the discrete direction given by the direction cosines \((\mu_i, \delta_i, \gamma_i)\), \( w_j \) is the angular quadrature weight associated with the incoming direction \( j \), \( \Phi_{ij} \) is the phase function for scattering between the discrete outgoing and incoming directions \( i \) and \( j \), \( I_c \) is the intensity due to a direct source, \( \Phi_{ic} \) is the phase function associated with this direct source, and the other quantities are defined in Chapter III. Details on inclusion of the direct beam \( I_c \) are given by Sánchez (1991). The total number of directions considered per octant is related to the quadrature order by \( K = N(N+2)/8 \).

To obtain a numerical solution of the partial differential equations given by Eq. (5.1), the domain is subdivided into a number of differential volumes \((\Delta V = \Delta x \Delta y \Delta z)\), and the differential equations are integrated over each volume. Using the finite-difference approximations

\[ \frac{\partial I_i^p}{\partial x} = \frac{I_i^c - I_i^w}{\Delta x}, \quad \frac{\partial I_i^p}{\partial y} = \frac{I_i^n - I_i^s}{\Delta y}, \quad \frac{\partial I_i^p}{\partial z} = \frac{I_i^f - I_i^b}{\Delta z} \]  

(5.2)

and relating the cell-mean and cell-edge intensities by

\[ I_i^p = \alpha I_i^x + (1-\alpha) I_i^e = \alpha I_i^y + (1-\alpha) I_i^r = \alpha I_i^z + (1-\alpha) I_i^f \]  

(5.3)

the discrete ordinates solution for the cell-mean intensity in the discrete direction \( i \) becomes

\[ I_i^p = \frac{\mu_i I_i^r}{\Delta x} + \frac{\delta_i I_i^r}{\Delta y} + \frac{\gamma_i I_i^r}{\Delta z} + \alpha \frac{S_i^p}{\Delta x + \Delta y + \Delta z} \]  

(5.4)

where the source term is given by

\[ S_i^p = k_a I_b + \frac{k_s}{4\pi} \sum_{j=1}^{8K} w_j I_j \Phi_{ij} + I_c \Phi_{ic} \]  

(5.5)
In Eqs. (5.3) and (5.4), superscripts "r" and "e" indicate reference (energy origination) and end-face (energy destination) values, superscript "p" denotes the average value over the volume, and \( \alpha \) is a finite-difference weighting factor. Equation (5.4) is subject to the boundary conditions

\[
I^+ = \varepsilon_o I_0^+ + \varepsilon_o \left[ I_{b_a}^0 + \delta(\mu_i - \mu_c) \delta(\delta_i - \delta_c) \delta(\gamma_i - \gamma_c) I_c^0 \right] \\
+ \frac{\rho_o}{\pi} f_d \left[ I_c^0 + \sum_{\nu_i<0} I_{\nu_i} \right] + \rho_o (1 - f_d) I_m
\]  

(5.6)

where the various quantities are defined in Section 4.4. To solve for the radiation field, Eq. (5.4) must be solved for each control volume. Because the source terms given by Eq. (5.5) and the boundary conditions given by Eq. (5.6) depend on the intensity distribution, an iterative solution is necessary.

5.1.2 Special Considerations

Because the DOM is a numerical solution to the equation of transfer, care must be taken to ensure stability of the procedure. In particular, it is important to understand how the quadrature set \((\mu_i, \delta_i, \gamma_i)\) and finite-difference weighting factor \(\alpha\) affect the solution. For example, it is known that a poor choice of \(\alpha\) may lead to unrealistic negative intensities. If \(\alpha = 1/2\), the resulting scheme is known as the diamond difference method, which is second order accurate. A choice of \(\alpha = 1\) results in the step method, which is first order accurate. Fiveland (1988) shows that, for the x-direction, the DOM is stable when

\[
\Delta x < \frac{\mu_i}{k_e (1 - \alpha) \zeta} 
\]  

(5.7)

where

\[
\zeta = \frac{a^3 + (1 - \alpha)^2 (2 - 5\alpha)}{\alpha}
\]  

(5.8)

Stability expressions for the y- and z-directions are obtained by exchanging \(\Delta y\) and \(\Delta z\) with \(\Delta x\), and \(\delta_i\), and \(\delta_j\) with \(\mu_i\), in Eq. (5.7). Examination of Eq. (5.7) reveals that the first order accurate step method is unconditionally stable, while the second order accurate diamond difference method guarantees stability only when the grid spacing in all directions approaches zero. As noted by Sánchez et al. (1992) and Roberti et al. (1993), the choice of
\( \alpha = 1/2 \) works well for one-dimensional problems, and for three-dimensional problems, \( \alpha = 1 \) is preferred to eliminate excessive CPU and memory requirements.

Finally, a number of researchers have found that the selection of DOM quadrature sets are very important for obtaining accurate solutions (Sánchez et al., 1992; Ehlert, 1992). For example, Sánchez (1991) suggested that if the phase function has been expanded in a Legendre polynomial representation (Appendix D), it should be renormalized for each discrete direction \( i \) according to

\[
\Phi_{11,ij} = \Phi_{11,ij} \frac{1}{8K} \sum_{j=1}^{4\pi} \Phi_{11,ij} w_j
\]

in order to satisfy the normalization requirement given by Eq. (3.32). This is because the coefficients of the Legendre expansion are computed using discrete values of scattering angle, which, for the DOM, depend on the choice of quadrature set. For additional details on choosing DOM quadrature sets, see Fiveland (1991), and Koch et al. (1995).

5.1.3 Extension of the DOM to solve the VRTE

To extend the DOM to solve the VRTE, the expanded form of the first row of Eq. (3.26) is given by

\[
\frac{dI_i}{ds} = -[k e_{11} I_i + k e_{12} Q_i + k e_{13} U_i + k e_{14} V_i] + k_{a1} I_b \\
+ \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 [Z_{11} I_i + Z_{12} Q_i + Z_{13} U_i + Z_{14} V_i] d\mu' d\phi'
\]

(5.10)

Analogous to the derivation of the DOM for unpolarized radiation, the derivation of the VDOM begins by replacing Eq. (5.10) with the set of partial differential equations for direction \( i \)

\[
\mu_i \frac{\partial I_i}{\partial x} + \delta_i \frac{\partial I_i}{\partial y} + \gamma_i \frac{\partial I_i}{\partial z} = -[k e_{11,i} I_i + k e_{12,i} Q_i + k e_{13,i} U_i + k e_{14,i} V_i] + k_{a1,i} I_b \\
+ \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{11,ij} I_j + Z_{12,ij} Q_j + Z_{13,ij} U_j + Z_{14,ij} V_j] w_j
\]
$$+ \frac{1}{4\pi} [Z_{11,i,c} I_c + Z_{12,i,c} Q_c + Z_{13,i,c} U_c + Z_{14,i,c} V_c]$$

(5.11)

where it should be emphasized that the extinction coefficients $k_e$ and absorption coefficient $k_a$ are dependent on the discrete direction $i$. As with the DOM, a numerical solution of Eq. (5.11) is obtained by subdividing the domain into a number of differential volumes $\Delta V = \Delta x \Delta y \Delta z$, and the differential equations are integrated over each volume. The partial derivatives can be rewritten using the finite-difference approximations

$$\frac{\partial I_i^p}{\partial x} \approx \frac{I_i^c - I_i^w}{\Delta x}, \quad \frac{\partial I_i^p}{\partial y} \approx \frac{I_i^n - I_i^s}{\Delta y}, \quad \frac{\partial I_i^p}{\partial z} \approx \frac{I_i^f - I_i^b}{\Delta z}$$

(5.12)

The cell-mean and cell-edge intensities can be assumed to vary as

$$I_i^p = \alpha I_i^{xe} + (1 - \alpha) I_i^{yr} = \alpha I_i^{ye} + (1 - \alpha) I_i^{ze}$$

(5.13)

so that

$$I_i^{xe} = \frac{I_i^p - (1 - \alpha) I_i^{yr}}{\alpha}; \quad I_i^{ye} = \frac{I_i^p - (1 - \alpha) I_i^{ze}}{\alpha}; \quad I_i^{ze} = \frac{I_i^p - (1 - \alpha) I_i^{xr}}{\alpha}$$

(5.14)

Now, taking $I_i^c$ as $I_i^{xe}$ and $I_i^w$ as $I_i^{xr}$ in Eq. (5.14),

$$I_i^c - I_i^w = \frac{I_i^p - (1 - \alpha) I_i^{xr}}{\alpha} - \frac{\alpha I_i^{xr}}{\alpha} = \frac{I_i^p - I_i^{xr}}{\alpha}$$

(5.15a)

For the $y$- and $z$- directions, the corresponding expressions are

$$I_i^n - I_i^s = \frac{I_i^p - I_i^{yr}}{\alpha}; \quad I_i^f - I_i^b = \frac{I_i^p - I_i^{zar}}{\alpha}$$

(5.15b)

Now, substitution of Eq. (5.15) into Eq. (5.11) yields

$$\mu_i \frac{I_i^p - I_i^{xr}}{\alpha \Delta x} + \delta_i \frac{I_i^p - I_i^{yr}}{\alpha \Delta y} + \gamma_i \frac{I_i^p - I_i^{ zar}}{\alpha \Delta z} = -k_{e11,i} I_i^p + S_{i}^{p,1}$$

(5.16)
where the source term for the first Stokes parameter is given by

\[ S_{1i}^{p,I} = S_{2i}^{p,I} + S_{3i}^{p,I} + S_{4i}^{p,I} \]  \hspace{1cm} (5.17)

and

\[ S_{1i}^{p,I} = k_{a1,i} I_b \]  \hspace{1cm} (5.18)

\[ S_{2i}^{p,I} = \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{11,ij} I_j^P + Z_{12,ij} Q_j^P + Z_{13,ij} U_j^P + Z_{14,ij} V_j^P] w_j \]  \hspace{1cm} (5.19)

\[ S_{3i}^{p,I} = -[k_{e12,i} Q_i + k_{e13,i} U_i + k_{e14,i} V_i] \]  \hspace{1cm} (5.20)

\[ S_{4i}^{p,I} = \frac{1}{4\pi} [Z_{11,ic} I_c + Z_{12,ic} Q_c + Z_{13,ic} U_c + Z_{14,ic} V_c] \]  \hspace{1cm} (5.21)

are the source/sink terms due to thermal emission, scattering of polarized radiation, extinction of polarized radiation, and polarized collimated beam radiation. Dividing Eq. (5.16) through by \( \Delta V \) and solving for the cell-mean intensity gives

\[ I_i^P = \frac{\mu_i \Delta x}{\Delta x} + \frac{\delta_i \Delta y}{\Delta y} + \frac{\gamma_i \Delta z}{\Delta z} + \alpha S_{i}^{p,I} \]  \hspace{1cm} (5.22a)

By analogy, expressions for the other cell-mean Stokes parameters are written as

\[ Q_i^P = \frac{\mu_i \Delta x}{\Delta x} + \frac{\delta_i \Delta y}{\Delta y} + \frac{\gamma_i \Delta z}{\Delta z} + \alpha S_{i}^{p,Q} \]  \hspace{1cm} (5.22b)

\[ U_i^P = \frac{\mu_i \Delta x}{\Delta x} + \frac{\delta_i \Delta y}{\Delta y} + \frac{\gamma_i \Delta z}{\Delta z} + \alpha S_{i}^{p,U} \]  \hspace{1cm} (5.22c)
\[
V_i^p = \frac{\mu_i}{\Delta x} V_{ix}^r + \frac{\delta_i}{\Delta y} V_{iy}^r + \frac{\gamma_i}{\Delta z} V_{iz}^r + \alpha \text{ } S_i^{P,V} \\
\frac{\mu_i}{\Delta x} \frac{\delta_i}{\Delta y} \frac{\gamma_i}{\Delta z} + \alpha \text{ } k_{e44,i}
\]

where the corresponding source terms are given by

\[
S_i^{P,Q} = -[k_{e21,i} I_i + k_{e23,i} U_i + k_{e24,i} V_i] + k_{a2,i} I_b \\
+ \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{21,ij} I_j^P + Z_{22,ij} Q_j^P + Z_{23,ij} U_j^P + Z_{24,ij} V_j^P] w_j \\
+ \frac{1}{4\pi} [Z_{21,ic} I_c + Z_{22,ic} Q_c + Z_{23,ic} U_c + Z_{24,ic} V_c] 
\]

\[
S_i^{P,U} = -[k_{e31,i} I_i + k_{e32,i} Q_i + k_{e34,i} V_i] + k_{a3,i} I_b \\
+ \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{31,ij} I_j^P + Z_{32,ij} Q_j^P + Z_{33,ij} U_j^P + Z_{34,ij} V_j^P] w_j \\
+ \frac{1}{4\pi} [Z_{31,ic} I_c + Z_{32,ic} Q_c + Z_{33,ic} U_c + Z_{34,ic} V_c] 
\]

\[
S_i^{P,V} = -[k_{e41,i} I_i + k_{e42,i} Q_i + k_{e43,i} U_i] + k_{a4,i} I_b \\
+ \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{41,ij} I_j^P + Z_{42,ij} Q_j^P + Z_{43,ij} U_j^P + Z_{44,ij} V_j^P] w_j \\
+ \frac{1}{4\pi} [Z_{41,ic} I_c + Z_{42,ic} Q_c + Z_{43,ic} U_c + Z_{44,ic} V_c] 
\]

These source terms may also be written in the form of Eq. (5.17), but are not written explicitly here. In summary, Eq. (5.22) for the VDOM has the same form as the corresponding DOM expression, Eq. (5.4). Also, the phase function renormalization discussed in Section 5.1.2 for the scalar DOM must be applied to the VDOM for each discrete direction i according to
where the double-tilde overbar indicates renormalization of the matrix. In the remainder of this work, the renormalization of Eq. (5.24) is applied unless stated otherwise and the phase matrix and related quantities appear without the double-tilde overbar.

Boundary conditions for the first Stokes parameter are written by expanding Eq. (4.26) and replacing the integral over direction with a summation over discrete directions as

\[ I^+ = \varepsilon_1 T_b^+ + [T_{11} I_b^0 + T_{12} Q_b^0 + T_{13} U_b^0 + T_{14} V_b^0] \]

\[ + \delta(\mu_i - \mu_c) \delta(\delta_i - \delta_c) \delta(\gamma_i - \gamma_c) [T_{11} I_c^0 + T_{12} Q_c^0 + T_{13} U_c^0 + T_{14} V_c^0] \]

\[ + (1 - f_d) [R_{11} I_m^- + R_{12} Q_m^- + R_{13} U_m^- + R_{14} V_m^-] \]

\[ + \frac{f_d}{\pi} [R_{11} I_c^- + R_{12} Q_c^- + R_{13} U_c^- + R_{14} V_c^-] \]

\[ + \frac{f_d}{\pi} \sum_{\nu_i < 0} [R_{11} I_i^- + R_{12} Q_i^- + R_{13} U_i^- + R_{14} V_i^-] \nu_i \]  

where the various symbols are defined in the discussion following Eq. (4.26). Similar expressions can be written for the other Stokes parameters. In summary, Eq. (5.25) for the VDOM boundary conditions has the same form as the corresponding DOM expression, Eq. (5.6).

### 5.1.4 DOM / VDOM Enhancements

In the DOM implementation of Sánchez (1991), several techniques were developed to enhance the traditional DOM. These enhancements have also been included in the VDOM implementation described here, and are as follows:

1. **Periodic boundaries**: Intensities going out of a boundary for a given iteration are placed as intensities coming into the opposite boundary for the next iteration for the direction being modeled. This is oftentimes a useful technique to use for 2- or 3-D problems where boundary conditions may not be well defined.
2. Symmetric boundaries: intensities going out of a boundary for a given iteration are placed as intensities coming into the opposite boundary for the next iteration for the direction mirroring that being modeled. This is useful for applications where the radiation field is known to be symmetric across a given plane. Note that this should not be used with a collimated beam that renders the radiation field non-symmetric.

3. Modeling 1-, 2-, or 3-D problems with the same model: by setting $\gamma_i = 0$ for all $i$ for 2-D problems, and setting $\gamma_i = 0$ and $\mu_i = 0$ for all $i$ for 1-D problems, it is possible to use a 3-D code to solve lower dimensional problems.

4. Evaluating intensities toward a sensor: this is described in Section 5.3.5.

### 5.2 General Computational Considerations

In this section, some computational issues associated with the DOM and VDOM are discussed. In Section 5.2.1, estimates of the amount of computer memory needed by the DOM and VDOM, and possible ways in which this amount may be reduced, are given. Section 5.2.2 outlines a source term linearization method that improves both the speed and accuracy of the DOM, and Section 5.2.3 describes how this method is extended to the VDOM.

#### 5.2.1 Memory Requirements

For three-dimensional problems, the computational resources needed by the DOM and VDOM depend on the dimensions of the three-dimensional discretization and on the number of discrete directions chosen. For the DOM, estimates of the memory needed can be computed by considering that there are six (double precision floating point) arrays of dimension $\text{NELEM} \times \text{ND}$, where $\text{NELEM}$ is the number if unique elements in the system and $\text{ND}$ is the number of discrete directions under consideration. For example, if each location in the system contains a unique element, then $\text{NELEM} = \text{NX} \times \text{NY} \times \text{NZ}$, where $\text{NX}$, $\text{NY}$, and $\text{NZ}$ are the number of control volumes in the $x$-, $y$-, and $z$-directions. The six arrays for each control volume/direction are for the following intensities: 1) front-back ($I^{xe}_{l} - I^{xe}_{r}$), 2) left-right ($I^{xe}_{l} - I^{xe}_{r}$), 3) top-bottom ($I^{ye}_{t} - I^{ye}_{b}$), 4) source ($S^{p}_{i}$), 5) cell-mean-current ($I^{p}_{i, \text{new}}$), and 6) cell-mean-previous ($I^{p}_{i, \text{old}}$).

There are two possible ways to reduce the amount of memory needed in this implementation of the DOM. First, if $\alpha = 1$, then Eq. (5.3) reduces to
\[ I_1^p = I_1^{xe} = I_1^{ve} = I_1^{ze} \]  

(5.26)

This means that arrays 1), 2), 3), and 5) above may be replaced by a single array. The second way to reduce the amount of memory needed is by using a convergence test that does not depend on storing the previous intensity values, thus eliminating the necessity of array 6). In other words, if both of these memory reduction techniques are employed, it is possible to cut in half the memory needed for the intensity arrays. Though not stated explicitly, both of these techniques were used in the DOM implementation of Roberti et al. (1993).

For the VDOM, the amount of memory required for a given problem could be several orders of magnitude larger than that required for the DOM, due to two major factors: 1) the VDOM is a vector equation of length four, rather than a scalar equation; and 2) the scattering phase matrix, extinction matrix, and emission vector for the VDOM are, in general, directionally dependent quantities. Item 2) is most severe when no assumptions can be made about the scatterers in the system, so that the scattering phase matrix is non-zero throughout its \( 4 \times 4 \) elements, and each element is a function of incoming and outgoing directions, rather than being solely a function of the scattering angle (the angle between the incoming and outgoing directions) as in the scalar DOM. In this situation, the number of double precision words required just for the storage of the scattering phase matrix is \( 4 \times 4 \times \text{NELEM} \times \text{ND}^2 \). Thus, a system with a spatial discretization of \( 20 \times 20 \times 20 \) and using an S-8 (ND = 80) quadrature scheme would need about 6.5 gigabytes of memory just to store the scattering phase matrix information if each element in the domain is assumed to be unique.

To reduce this memory requirement, it may become necessary to make certain simplifying assumptions regarding the polarized radiative transfer domain. For example, by making an assumption of randomly oriented axially symmetric particles, the phase matrix can be reduced to six unique parameters that are a function of scattering angle, as described in Appendix D. In addition, for this situation, the extinction matrix becomes a scalar quantity independent of direction. It is also possible, of course, to compute scattering quantities during the VDOM solution procedure so that the entire directionally-dependent scattering phase matrix does not need to be stored; however, this would usually make the computational time prohibitive. Another way to effectively reduce memory requirements is to take advantage of parallel or distributed computing methodologies, as is discussed in Section 5.3.
5.2.2 DOM Source Term Linearization

The source term linearization technique for the DOM was developed by Chai et al. (1992) for one- and two-dimensional geometries, and is described here for application to three-dimensional geometries. Consider that the intensity $I_i$ being sought on the left hand side of Eq. (5.1) is also contained on the right hand side of this equation through the source term as expressed by Eq. (5.5). The linearization technique subtracts the forward scattering component from the scattering source term so that the intensity of interest is no longer contained by the scattering source function. In other words, Eq. (5.1) is rewritten as

$$\mu_i \frac{\partial I_i}{\partial x} + \delta_i \frac{\partial I_i}{\partial y} + \gamma_i \frac{\partial I_i}{\partial z} = -\tilde{k}_e I_i + \tilde{S}_i^p$$ \hspace{1cm} (5.27a)

where the modified extinction coefficient and modified scattering source term are given by

$$\tilde{k}_e = k_e - \frac{k_s}{4\pi} w_i \Phi_{ii}$$ \hspace{1cm} (5.27b)

$$\tilde{S}_i^p = k_a I_b + \frac{k_s}{4\pi} \sum_{j=1}^{8K} w_j I^p_{jj} \Phi_{ij} + \frac{k_s}{4\pi} I_c \Phi_{ic}$$ \hspace{1cm} (5.27c)

Notice that the modification of the extinction coefficient effectively reduces the optical thickness of the problem. Applying the DOM discretization procedure to Eq. (5.27) allows Eq. (5.4a) to be replaced with

$$I_i^p = \frac{\mu_i}{\Delta x} I_i^{x \tau} + \frac{\delta_i}{\Delta y} I_i^{y \tau} + \frac{\gamma_i}{\Delta z} I_i^{z \tau} + \alpha \tilde{S}_i^p$$ \hspace{1cm} (5.28)

$$\Delta x \Delta y \Delta z$$

For two-dimensional problems, Chai et al. (1992) reported CPU savings of up to 77% by using this solution technique.

5.2.3 VDOM Source Term Linearization

In this section, the source term linearization applied to the DOM in Section 5.2.1 is extended to the VDOM. The linearization derivation begins by rewriting the summation on the right hand side of Eq. (5.11) as
\[
\sum_{j=1}^{8K} [Z_{11,ij} I_j + Z_{12,ij} Q_j + Z_{13,ij} U_j + Z_{14,ij} V_j] w_j = \\
\sum_{j=1}^{8K} [Z_{11,ij} I_j + Z_{12,ij} Q_j + Z_{13,ij} U_j + Z_{14,ij} V_j] w_j \\
+ [Z_{11,ii} I_i + Z_{12,ii} Q_i + Z_{13,ii} U_i + Z_{14,ii} V_i] w_i \tag{5.29}
\]

Substitution of Eq. (5.29) in Eq. (5.11) produces

\[
\mu_i \frac{\partial I_i}{\partial x} + \delta_i \frac{\partial I_i}{\partial y} + \gamma_i \frac{\partial I_i}{\partial z} = I_i \left[ -k_{e11,i} + \frac{1}{4\pi} Z_{11,ii} \right] + Q_i \left[ -k_{e12,i} + \frac{1}{4\pi} Z_{12,ii} \right] \\
+ U_i \left[ -k_{e13,i} + \frac{1}{4\pi} Z_{13,ii} \right] + V_i \left[ -k_{e14,i} + \frac{1}{4\pi} Z_{14,ii} \right] + k_{a1,i} I_b \\
+ \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{11,ij} I_j + Z_{12,ij} Q_j + Z_{13,ij} U_j + Z_{14,ij} V_j] w_j \\
+ \frac{1}{4\pi} [Z_{11,ic} I_c + Z_{12,ic} Q_c + Z_{13,ic} U_c + Z_{14,ic} V_c] \tag{5.30}
\]

or

\[
\mu_i \frac{\partial I_i}{\partial x} + \delta_i \frac{\partial I_i}{\partial y} + \gamma_i \frac{\partial I_i}{\partial z} = -[\kappa_{e11,i} I_i + \kappa_{e12,i} Q_i + \kappa_{e13,i} U_i + \kappa_{e14,i} V_i] + k_{a1,i} I_b \\
+ \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{11,ij} I_j + Z_{12,ij} Q_j + Z_{13,ij} U_j + Z_{14,ij} V_j] w_j \\
+ \frac{1}{4\pi} [Z_{11,ic} I_c + Z_{12,ic} Q_c + Z_{13,ic} U_c + Z_{14,ic} V_c] \tag{5.31a}
\]

where the modified extinction coefficients are given by

\[
\kappa_{e1n,i} = \kappa_{e1n,i} - \frac{1}{4\pi} Z_{1n,ii}, \quad n = 1..4 \tag{5.31b}
\]
Notice that Eq. (5.31a) has the same form as Eq. (5.11). Thus, applying the procedure used in Eqs. (5.12) through (5.16) to Eq. (5.31) allows the first cell-mean Stokes parameter to be solved for as

\[
I_i^p = \frac{\mu_i I_i^{x_i} + \delta_i I_i^{y_i} + \gamma_i I_i^{z_i} + \alpha \xi_i^p \bar{k}_{e1,i}}{\Delta x \Delta y \Delta z + \alpha \bar{k}_{e11,i}}
\]  

(5.32)

where the modified source term is

\[
\xi_i^p = -[\bar{k}_{e12,i} Q_i + \bar{k}_{e13,i} U_i + \bar{k}_{e14,i} V_i] + k_{a1,i} I_b + \frac{1}{4\pi} \sum_{j=1}^{8K} [Z_{11,ij} I_j^p + Z_{12,ij} Q_j^p + Z_{13,ij} U_j^p + Z_{14,ij} V_j^p] w_j
\]

\[
+ \frac{1}{4\pi} [Z_{11,ic} I_c + Z_{12,ic} Q_c + Z_{13,ic} U_c + Z_{14,ic} V_c]
\]

(5.33)

Expressions for the other three cell-mean Stokes parameters are obtained by analogy, and are not written here. In addition, the modified source terms can be written as in Eq. (5.17). In summary, Eq. (5.32) is of the same form as Eq. (5.22a). Thus, the general VDOM solution procedure does not change when the source term linearization is applied, and, the modified extinction coefficient defined in Eq. (5.31b) effectively reduces the optical thickness of the problem.

### 5.3 VDOM Distributed Computing Methodology

Because the implementation of the DOM is computationally resource intensive, as alluded to above, the utilization of a parallel computing methodology is sought. Successful applications of parallel computing in the solution of two-dimensional radiative transfer problems are described by Hanebutte and Lewis (1991), and Saltiel and Naraghi (1993). For 1-, 2-, and 3-D radiative transfer problems, Haferman et al. (1994b) use a parallel implementation of the DOM that assigns sets of discrete directions to a set of processors on a massively parallel computer.

In this section, the implementation of the VDOM using a distributed network of computer processors is described. An overview of the implementation is presented in
Section 5.3.1. Section 5.3.2 details the computation of boundary conditions using the distributed network, and Section 5.3.3 discusses the inclusion of a collimated beam in the model. Finally, Section 5.3.4 outlines how the Stokes parameters for user-specified extra-quadrature directions are computed.

5.3.1 Overview

The VRTE given by Eq. (3.26) consists of $4 \times ND$ coupled equations, each with an identical form; thus, each equation may be given to a separate CPU or set of CPUs. However, because the equations are coupled, the processing nodes must communicate their partial results to the other nodes before obtaining a complete result. The approach taken here is to distribute the equations for a given Stokes parameter to a single processing unit, so that each unit considers ND discrete directions. In essence, each unit solves a set of equations similar to the scalar DOM. A schematic representation of this distributed computational solution technique for the VDOM is shown in Figure 5.1.

An existing software package called Parallel Virtual Machine (PVM) (Geist et al., 1994), allows a collection of computers to function as a parallel computer. Currently, PVM is the de facto standard for distributed computing applications, though, recently, the parallel programming community has made an effort to adopt the Message Passing Interface (MPI) (Gropp et al., 1994) as a standard. However, MPI is currently not as widely available as PVM, and thus, the latter is used for implementing the VDOM. In theory, the distribution of tasks across a set of processing units may have the benefit of improving execution times relative to those obtained using a single processing unit. In addition, computer memory usage is split across the set of processors. A benefit of using a package such as PVM or MPI is that the package offers the flexibility of tailoring the network of distributed machines to what the researcher has available: for example, the packages can be run on a heterogeneous network, or, they may be configured to run on a single processor unit that effectively simulates several processors. Additional comments are given by Geist et al. (1994) and Gropp et al. (1994).

The flow of the top level processes for the parallel implementation of the VDOM for the first Stokes parameter is shown in Fig. 5.2. This flowchart represents the box marked "I" in Fig. 5.1, though the same flow takes place for each of the other Stokes parameters. The ovals designated $I$, $Q$, $U$, and $V$ in Fig. 5.2 indicate where data are exchanged with other processors. As depicted in Fig. 5.2, the first iteration in the VDOM solution procedure begins with an initialization and then Eq. (5.22a) is solved. For the first iteration, the source term $S_{i}^{P,J}$ is set to zero. If the solution is not converged, the scattering
Figure 5.1 Schematic representation of distributed computational solution technique for the VDOM. The four Stokes parameters I, Q, U, and V are each assigned to a processing unit. Each processing unit solves for ND discrete directions. The Stokes parameters are coupled through the VRTE, so that processors must exchange information with one another, as represented by the oval designated "control."
Figure 5.2 VDOM top level process flowchart. This chart is a schematic of the box marked "I" in Fig. 5.1 and represents the computational flow for the computer processor solving for the I Stokes parameter. The same computational flow takes place on the processors solving for the Q, U, and V Stokes parameters. The ovals designated I, Q, U, V indicate where data are exchanged with the other processors.
source term given by Eq. (5.19) is computed, including the collimated beam contribution of Eq. (5.21) if necessary. The computation of the scattering source term requires that all processing nodes communicate with one another. For example, the node that computes I needs the values of Q, U, and V. After the scattering source term is computed, the boundary conditions are updated using Eq. (5.25). Once again, all processing nodes must exchange information with each other for this step. Finally, after the boundary conditions are computed, the extinction source term is computed using Eq. (5.20), and the next iteration proceeds. After convergence is obtained, the Stokes parameters are solved for any user-desired directions (described in Section 5.3.5), and then the results are printed.

Because of memory requirements, the values of I, Q, U, and V are exchanged between processors three times during a given iteration: 1) for the computation of the scattering source term; 2) for the computation of the boundary conditions; and 3) for the computation of the extinction source term. For example, when the scattering source term is computed on the I Stokes parameter node, the summation of Eq. (5.19) is performed using the local value of I. Then, the array that holds the value of I is overwritten with the values of Q, and the terms containing Q in Eq. (5.19) are accumulated, and so forth with the values of U and V. Because the values of I are written over, they must be restored before the extinction source is computed (the boundary intensities are not written over, and thus I does not need to be restored before computation of the boundary conditions). The procedure is written algorithmically in Fig. 5.3.

```
call initialize()
call solve()
while (not (converged)) {
    call scatter_pvm()
call boundary_pvm()
call restore()
call extinct_source_pvm()
call solve()
call check_convergence()
}
call user_angles()
call printout()
exit()
```

Figure 5.3 VDOM top level algorithm.
5.3.2 Source Terms

The source terms are due to emission, scattering of diffuse radiation, scattering of collimated (direct) radiation, and polarized extinction, as described by Eq. (5.17). For the computer implementation of the VDOM, the emission source term, given by Eq. (5.18), is straightforward to compute because there are no cross-polarization contributions (for example, the processor that is computing the I Stokes parameter does not need any information from the Q, U, or V processors). However, the other source terms due contain cross-polarization contributions.

The scattering source term is computed using the algorithm depicted in Fig. 5.4. On each node, the scattering source term is zeroed out, and the contribution of the local node ("me" in Fig. 5.4) to Eq. (5.19) is computed. Then, the local intensities are exchanged with the other processors ("iwho" in Fig. 5.4), and as the intensities are received from the other nodes, their contribution to Eq. (5.19) is accumulated. The computation of the extinction source term of Eq. (5.20) is very similar, except that the local nodes make no contribution to the term. The algorithm for the extinction source term is shown in Fig. 5.5.

```
call zero_scattering_term()
call compute_scattering_term(me)
call broadcast_intensities(me)
do (iwho = 1,4) {
    if (iwho ≠ me) {
        call receive_intensities(iwho)
call compute_scattering_term(iwho)
    }
}
return()
```

Figure 5.4 VDOM algorithm for scattering source term.
call broadcast_intensities(me)
do (iwho = 1,4) {
    if (iwho ≠ me) {
        call receive_intensities(iwho)
call compute_extinction_term(iwho)
    }
}
return()

Figure 5.5 VDOM algorithm for extinction source term.

5.3.3 Boundary Conditions

The implementation of the boundary conditions for the VDOM is shown algorithmically in Fig. 5.6. The algorithm begins by computing the boundary conditions of Eq. (5.25) for the local node. Then, to reduce communication and memory overhead, the boundary intensity arrays are packed into the cell-mean intensity array and broadcast to the other processing nodes. After this step, the local node receives (into the cell-mean intensity array) the boundary condition intensities sent from the other processors, and unpacks these values into the local boundary condition array. Then, Eq. (5.25) is applied to this updated array. Finally, local boundary intensities, which were written over during the packing/unpacking procedure, are restored.

call boundary(me)
call pack_intensities(me)
call broadcast_intensities(me)
do (iwho = 1,4) {
    if (iwho ≠ me) {
        call receive_intensities(iwho)
call unpack_intensities(iwho)
call boundary(iwho)
call boundary_restore()
    }
}
return()

Figure 5.6 VDOM algorithm for boundary conditions.
5.3.4 Implementation of Parallel Beam

The source term for scattering of the parallel beam, given by Eq. (5.21), is of the same form as the equation for the scattering of diffuse radiation, given by Eq. (5.19). Thus, the computer implementation of the parallel beam is straightforward, and is computed in the same subroutine as the (5.19). That is, in the routine compute_scattering_term() depicted in Fig. 5.4, the parallel beam term is added in if it exists. In addition, because the beam is assumed to be unpolarized, only \( I_C \) is non-zero. The only complication is that, at the beginning of the solution procedure, the intensity of the unpolarized collimated source is known only at the boundary at which it is entering. But, for Eq. (5.19), the intensity of the collimated source is needed at all discrete spatial locations within the numerical solution domain. As noted by Sánchez (1991), the collimated intensity entering the solution domain is exponentially attenuated along its path of travel. So, using a ray-tracing technique, it is possible to compute the intensity \( I_C \) at any position within the spatial domain. For further details on the numerical implementation of the collimated source, see Sánchez (1991).

5.3.5 Solving for the Stokes Vector in a User-Specified Direction

As first noted by Sánchez (1991), it is possible, using the DOM, to compute intensities at any desired outgoing direction. This is because, after applying the DOM solution procedure, the discrete intensities are known throughout the spatial domain for each discrete direction. For the VDOM, the procedure to compute the Stokes parameters in any desired user-specified direction is completely analogous to the DOM procedure and is as follows:

1. The boundary conditions in the direction of the sensor are calculated using Eq. (5.25) and replacing subscript "i" with "s" (for "sensor").

2. The source terms in the sensor direction are evaluated using Eq. (5.17) and replacing subscript "i" with "s".

3. Eqs. (5.22) and (5.14) are applied in the sensor direction to obtain the desired intensities.

The only major complication is that the phase matrix is required for each combination of VDOM discrete direction and user-specified direction. However, for the current implementation, a post-processing procedure is used that computes the phase matrix "on-the-fly" using spherical functions. This procedure does not work when a specularily
reflecting surface is used due to the unavailability of the intensity mirroring the sensor direction that is required in Eq. (5.25).

5.4 Summary
In this chapter, the DOM used to solve the RTE has been reviewed, and the VDOM for solving the VRTE has been derived. The general VDOM solution procedure has been discussed, including various computational considerations: for example, a description of the VDOM solution procedure using a distributed computing approach was outlined. In addition, implementation of a collimated source, a VDOM source term linearization technique, boundary condition implementation details, and evaluation of output intensities in user-specified directions were discussed. In the subsequent chapters, the VDOM is tested and applied to various problems.
CHAPTER VI

VALIDATION OF MODEL

The first several sections of this chapter validate the VDOM for polarized radiation by comparing its results to benchmark 1-D cases available in the literature, namely, those presented by Evans and Stephens (1991) and Wauben and Hovenier (1992). For Tests 1 to 8, randomly oriented symmetric particles are considered, so that the extinction matrix is diagonal and consists of one element, and the scattering phase matrix consists of four elements, as detailed in Appendix D. Discussion of the applicability of the VDOM to oriented particles situations is given in Section 6.9. For 2-D polarized radiative transfer, only one case is available in the literature (Jin, 1991), and the inputs to that model are not specified so that no validation can be performed for the VDOM for this type of geometry. There are no examples of 3-D polarized radiative transfer in the literature, precluding a validation of the VDOM for that type of geometry. However, the final section of the chapter describes tests of the VDOM for 2- and 3-D systems that establish confidence that the model functions properly for these dimensions.

6.1 Test 1: No Atmosphere, Specular Surface

Test 1 is based on the "No atmosphere" results in Table 8 of Evans and Stephens (1991).

6.1.1 Description

A schematic for Test 1 is given in Fig. 6.1. The objective of this case is to compute, for a frequency of 85.5 GHz, the upwelling radiation from calm water modeled as a specular surface at a temperature of 300 K with a complex refractive index of 3.724 – 2.212i. This is a convenient case because, for no atmosphere, the medium is non-participating so that all terms on the right hand side of Eq. (3.26) vanish. Thus, each upwelling Stokes parameter is due solely to radiation emitted by the surface, and is a position independent constant along a given line-of-sight. Mathematically, this is stated using the boundary condition given by Eq. (4.26), where the first term on the right hand side accounts for the surface emission. Further inspection of Eqs. (3.26) and (4.26) reveals that the Stokes parameters are uncoupled for this case and that U and V are identically zero due to Eqs. (4.26) and (4.32). Thus, in terms of brightness temperature, the exact solution for this case is given by

65
Figure 6.1 Schematic for Test 1.

\[ \bar{T}_B = \bar{\varepsilon}^0 T \]  \hspace{1cm} (6.1)

where \( \bar{T}_B \) is the brightness temperature vector associated with the Stokes vector, \( \bar{\varepsilon}^0 \) is the surface emissivity, which, for this case, is given by Eq. (4.28), and \( T \) is the surface temperature.

6.1.2 Results

The results for Test 1 for the I and Q Stokes parameters are computed using the VDOM and are converted to brightness temperatures using Eq. (3.41). The results of these computations are depicted in the plots of Fig. 6.2. Values of \( T_{B,I} \) and \( T_{B,Q} \) are plotted in the figure as a function of zenith viewing angle \( \mu \), and are generated using the sixth order level symmetric odd (LSO-6) and eighth order equal weight odd (EWO-8) quadrature sets (Fiveland, 1991). The exact solution is computed using Eq. (6.1), and the user-angle results are computed using the EWO-8 quadrature set results with the procedure outlined in Section 5.3.5. The user-angles for this test are \( \mu = 0.09501, 0.28160, 0.45802, 0.61788, 0.75540, 0.86563, 0.94458, \) and \( 0.98940 \), which are chosen for comparison with the eight-angle Gaussian quadrature used by the RT3 polarized radiative transfer code of Evans and Stephens (1991), and are used in some of the subsequent tests presented in this chapter.
The results of the VDOM for $T_{B,1}$ and $T_{B,O}$ are computed using $NY = 10$ vertical layers, and are in excellent agreement with the exact solution.

Figure 6.2 Results of Test 1.
6.2 Test 2: Homogeneous Isothermal Non-Scattering Absorbing Atmosphere, Specular Surface

Test 2 is based on the "upwelling" and "downwelling" results in Table 8 of Evans and Stephens (1991), but excludes background radiation and replaces their ice/rain atmosphere with an isothermal, non-scattering, absorbing atmosphere.

6.2.1 Description

A schematic for Test 2 is presented in Fig. 6.3. This test uses the same specularly reflecting surface as in Test 1, and places a 8 km thick, homogeneous, non-scattering, absorbing atmosphere above this surface. The atmosphere is taken to be isothermal with a temperature of 300 K. For this situation, the extinction matrix is diagonal and consists of one parameter \( k_e \) (see Appendix D.3.1), and, because there is no scattering, the emission vector has \( k_{a1} = k_e = 0.01 \text{ km}^{-1} \), and its other elements are equal to zero, according to Eq. (4.25). As noted by Evans and Stephens (1991), an azimuthally symmetric radiation field (for example, a plane-parallel atmosphere with only thermal sources, as in this case) has zero U and V components for the Stokes vector. Thus, all matrices in the VRTE reduce to \( 2 \times 2 \), and only solutions for the I and Q Stokes parameters are sought. An exact, analytic solution for this case can be derived, and is given in terms of the Rayleigh-Jeans brightness temperature approximation in Appendix D.4. Examination of the VRTE and associated boundary conditions for this case show that polarization (that is, a non-zero Q Stokes parameter) is due solely to emission and specular reflection by the surface.

![Schematic for Test 2](image)

**Figure 6.3** Schematic for Test 2.
6.2.2 Results

The results for this Test are computed for the I and Q Stokes parameters using the VDOM and are converted to brightness temperatures using the Rayleigh-Jeans approximation of Eq. (3.42) at a frequency of 85.5 GHz. The VDOM results are computed using NY = 10 layers, and are plotted in Fig. 6.4. Values of upwelling $T_{B,I}$ and $T_{B,Q}$ at $y = 8$ km, and downwelling $T_{B,I}$ at $y = 0$ km, are plotted in the figure as a function of $\mu$ as generated using the LSO-6 quadrature set. The exact solution is computed using Eq. (D.34), and the values of the user-angles are as for Test 1, but are generated using the results of the LSO-6 quadrature. For this case, the downwelling $T_{B,Q}$ at $y = 0$ km is zero for all viewing angles, so this result is not shown in the figure. Each of the plots in Fig. 6.4 demonstrate that the VDOM results agree very well with the exact solution.

6.3 Test 3: Two-Layer Precipitating Atmosphere, Specular Surface

Test 3 is based on the "upwelling" and "downwelling" results in Table 8 of Evans and Stephens (1991).

6.3.1 Description

A schematic for Test 3 is presented in Fig. 6.5. This test uses the same specularly reflecting surface as in Tests 1 and 2, but places an 8 km thick, two-layer precipitating atmosphere above the surface. The lower half of this atmosphere is modeled as rain and the upper half is taken as ice particles. The atmosphere has a temperature profile that varies linearly within each layer such that $T = 300$ K at the surface, $T = 273$ K at the ice/rain interface, and $T = 245$ K at $y = 8$ km. Both layers have Marshall-Palmer dropsizes distributions of spherical particles (Eq. (4.19)), and the rainrate within the rain layer is 0.5 mm/h and within the ice layer is 2.0 mm/h. The maximum particle diameter within each layer is 1 cm. Unpolarized, diffuse, background radiation at a blackbody equivalent temperature of 2.7 K, representative of realistic cosmic background radiation, is incident from above the atmosphere. Because the particles in this test case are spherical, the extinction matrix for each layer is diagonal and consists of one parameter $k_e$, and the phase matrices for each layer has the form given in Appendix D.2.2. Because the atmosphere is plane-parallel with only thermal sources of radiation, U and V are equal to zero everywhere, so that all matrices in the VRTE reduce to $2 \times 2$. The values of the radiative properties for each layer have been computed by Evans and Stephens (1991) and the
Figure 6.4 Results of Test 2.
Figure 6.5 Schematic for Test 3.
extinction coefficient and single scattering albedo (defined in Eq. (4.17)) are specified in Fig. 6.5. The parameters $P_1$, $P_2$, and $P_3$ needed for the $2 \times 2$ version of the phase matrix of Eq. (D.8) are each expressed using the Legendre series expansion of Eq. (D.26), where the expansion coefficients are listed in Table 6.1. It should be pointed out that applying Eq. (D.26) to these expansion coefficients yields the elements of the normalized Mueller matrix $\mathbf{\Phi}$; this matrix must be multiplied by $\mathbf{w}$ to obtain the phase matrix $\mathbf{F}$ needed for the VDOM. Polarization of the radiant energy (a non-zero value of $Q$) in this case is due to scattering by the spherical particles as well as emission and specular reflection by the surface. The frequency under consideration is 85.5 GHz.


<table>
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<tr>
<th></th>
<th>Rain</th>
<th>Ice</th>
</tr>
</thead>
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<td>$\alpha_2^1$</td>
</tr>
<tr>
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<td>-0.378560</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
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</tr>
<tr>
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<td>0.000819</td>
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<td>9</td>
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<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6.3.2 Results

Values of upwelling and downwelling $T_{B,I}$ and $T_{B,Q}$ at $y = 0$ and 8 km are plotted in Fig. 6.6 as a function of $\mu$ as generated by the VDOM using the LSO-6 quadrature set and the Rayleigh-Jeans brightness temperature approximation. Also plotted in the figure
Figure 6.6 Results of Test 3.
are the results for the same case generated by the RT3 polarized radiative transfer code of Evans and Stephens (1991) (labeled "Evans" in the figure). The RT3 code has been extensively validated, and its accuracy increases as the number of discrete angles used in the quadrature increases. For the comparison in this chapter, an eight-angle Gaussian quadrature is used, which should give reasonably high accuracy.

The plots in Fig. 6.6 indicate that the VDOM results compare favorably with the RT3 results. User-angle generation, as described in Section 5.3.5, is not implemented for systems with specularly reflecting surfaces, so that a direct comparison of the VDOM results with those of Evans and Stephens is not possible for this case. However, it appears that the largest difference between the VDOM and RT3 results are for upwelling $T_{B,1}$ at $y = 8 \text{ km}$, where the VDOM results seem slightly low. These VDOM results are computed using $NY = 10$, and change less than a factor of $10^{-4}$ by increasing $NY$ to 20. However, using more angles (LSO-8; results not shown) produces a change of up to a few per cent depending on the Stokes parameter and value of $\mu$ under consideration, and, for this case, gives excellent agreement with the RT3 results. A comparison of Fig. 6.6 with Fig. 6.2 indicates that the values of $T_{B,Q}$ are much smaller in magnitude for the atmosphere containing spherical rain and ice particles than for the "no atmosphere" case. Qualitatively, this means that the spherical ice and rain tend to depolarize the radiant energy emitted and reflected by the underlying specular surface.

6.4 Test 4: Two-Layer Precipitating Atmosphere, Diffuse Surface

Test 4 is based on the "upwelling" and "downwelling" results in Table 8 of Evans and Stephens (1991), but uses a diffuse surface.

6.4.1 Description

The system used in Test 4 is identical to that used for Test 3, except that the surface is modeled as a diffuse surface, as shown schematically in Fig. 6.7. In this case, for the boundary condition expression of Eq. (4.26), $f_d = 1.0$, signifying that the reflected energy from the surface is diffuse. The reflection matrix and emission vector appropriate for this case are those given by Eqs. (4.27) and (4.28), where $\rho_0$ is set to 0.44, and $\varepsilon_0 = 1 - \rho_0 = 0.56$. Because a diffuse surface does not contribute to polarization of radiant energy, any polarized signal for this situation (non-zero value of $Q$) must be due to scattering by the spherical rain and ice particles.
Figure 6.7 Schematic for Test 4.
6.4.2 Results

VDOM results for upwelling and downwelling $T_{B, I}$ and $T_{B, Q}$ at $y = 0$ and 8 km are plotted in Fig. 6.8 as a function of $\mu$. These results are for the LSO-6 quadrature set and the Rayleigh-Jeans brightness temperature approximation at a frequency of 85.5 GHz. Plotted in the same figure are eight-angle Gaussian quadrature results of the RT3 polarized radiative transfer code (labeled "Evans" in the figure). The VDOM results always fall within 0.1 K of the straight line produced by connecting the points of the RT3 results. As in Test 3, the VDOM results for this case are insensitive to the vertical spatial discretization: NY = 10 results are plotted in Fig. 6.8, an increase to NY = 20 yielded insignificant differences in the $T_B$ values. Use of the LSO-8 quadrature (results not shown) appears to give even better agreement with the RT3 values.

In contrast to the results of the Test 3, the results of this case demonstrate that the spherical rain and ice particles slightly polarize the radiation that is emitted and reflected by the underlying diffuse water surface. Also, the plot for upwelling $T_{B, Q}$ at $y = 8$ km has points less than zero for $\mu$ less than about 0.2 and $\mu$ near 1.0. Referring to Eqs. (3.19) and (3.20), values of $Q$ less than zero indicate that the horizontally polarized intensity is greater than the vertically polarized intensity. However, the degree of polarization, defined in Eq. (3.10), is less than about 0.01 for all values of $\mu$ plotted in Fig. 6.8.

6.5 Test 5: Mie Scattering of Collimated Beam, Diffuse Surface

Test 5 is based on the "Mie scattering test case" of Evans and Stephens (1991).

6.5.1 Description

A schematic for Tests 5 to 8 is given in Fig. 6.9. For these tests, a cold ($T = 0$ K) atmosphere of optical thickness 1.0 is irradiated from above by an unpolarized collimated beam of intensity $I_c$ at a zenith angle of $\theta_c$ and an azimuthal angle of 0 deg. The surface is diffuse with a reflectance of $\rho_o$, an emittance of $\varepsilon_o = 1 - \rho_o$, and a temperature of 0 K. The atmosphere absorbs, scatters, and emits radiant energy, and is specified for the VRT in terms of a phase matrix and single scattering albedo, as well as the frequency under consideration. For these tests, the extinction matrix is diagonal, and $k_e$ is set to unity. For each test, the atmosphere is assumed to consist of an ensemble of spheroidal particles characterized by the ratio $a/b$, where $a$ and $b$ are the lengths of the semi-major and -minor axis of the spheroids. Table 6.2 gives the values of these parameters, and gives details on the spheroid size distribution for Tests 5 to 8. It is emphasized that, for these tests, the
Figure 6.8 Results of Test 4.
Figure 6.9 Schematic for Tests 5 to 8.
Table 6.2 Parameters for Tests 5 to 8. $I_c$ is the intensity of the collimated beam, $\mu_c$ is the cosine of the beam zenith angle, $\rho_o$ is the diffuse surface reflectance, $\lambda$ is the wavelength, $\Theta$ is the single-scattering albedo of the particles, and a and b are the lengths of the semi-major and -minor axes of the spheroids. $I_c$ has units of $W \cdot m^{-2} \cdot \mu m^{-1} \cdot sr^{-1}$, $\lambda$ has units of $\mu m$, and all other quantities are dimensionless. The azimuthal angle of the incoming collimated beam is $\phi_c = 0$ deg.

<table>
<thead>
<tr>
<th>CASE</th>
<th>$I_c$</th>
<th>$\mu_c$</th>
<th>$\rho_o$</th>
<th>$\lambda$</th>
<th>$\Theta$</th>
<th>a/b</th>
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<tr>
<td>Test 5</td>
<td>$\pi$</td>
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<td>0.1</td>
<td>0.951</td>
<td>0.99</td>
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<td></td>
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</tr>
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<td>Test 6</td>
<td>$\pi$</td>
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<td>4.0</td>
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<tr>
<td>comments:</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>Test 7</td>
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<td></td>
<td></td>
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<td>Test 8</td>
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<td></td>
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</tr>
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unpolarized collimated beam is the only source of radiant energy, and scattering by the spheroidal particles is the only contributor to polarization of this energy. In addition, in contrast to the previous tests, the collimated beam renders the radiation field asymmetric, so that, in general, the U and V Stokes parameters are non-zero for these tests.

For Test 5, the elements of the normalized Mueller matrix $\overline{\Psi}$ are computed using Eq. (D.26) and the Legendre series expansion coefficients listed in Table 6.3. The normalized Mueller matrix is then multiplied by $\Phi$ to obtain the phase matrix $\overline{\Phi}$ needed for the VDOM.

Table 6.3 Legendre series coefficients of the phase matrix elements for Test 5. From Evans and Stephens (1991).

<table>
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6.5.2 Results

Some results of Test 5 are given in Figs. 6.10 to 6.12 and Tables 6.4 and 6.5. As mentioned in Section 6.5.1, the radiation field for this test problem is asymmetric due to the azimuthal directional component of the collimated beam. Thus, the results presented in the figures and tables for this test are given as a function of \( \mu \) and \( \phi \), and, in particular, \( \mu \) ranges from 0.0 to 1.0 depending on the quadrature set and user-angles chosen for a particular set of results, and \( \phi = 0, 90, \) or 180 deg. These values of \( \phi \) are used for comparison with the results published by Evans and Stephens (1991). Also, \( U \) and \( V \) are zero for \( \phi = 0 \) and 180 deg, so results for \( U \) and \( V \) are not presented for these values of \( \phi \).

The plots in Fig. 6.10 show values of \( I \) and \( Q \) upwelling at the top of the atmosphere as a function of \( \mu \) (for \( \phi = 0 \) deg) as obtained by the RT3 model of Evans and Stephens (1991), and the VDOM. The RT3 results are generated using a sixteen angle Gaussian quadrature, and the VDOM values are obtained using the EWO-6 and EWO-8 quadratures to generate user-angle results at the values of \( \mu \) that are given in Section 6.1.2. The RT3 results were validated by Evans and Stephens (1991) against another polarized radiative transfer model, and were on the average within about 0.25 per cent of the values from that model. The results plotted in Fig. 6.10 indicate a good agreement between RT3 and VDOM for \( I \), though the VDOM value obtained using the EWO-6 quadrature \( \mu = 0.09501 \) appears to be slightly low. Better agreement between RT3 and the VDOM is obtained using the EWO-8 quadrature. The plot of \( Q \) versus \( \mu \) in Fig. 6.10 indicates that, though the VDOM and RT3 curves have the same shape, the VDOM results tend to fall to either side of the RT3 line. The disagreement between the results from the two models appears to increase as \( \mu \) decreases. Evans and Stephens (1991) noted that discrete angle radiative transfer solution methods typically give poor results at small values of \( \mu \), a problem that can generally be remedied by increasing the number of discrete angles. This notion seems to be supported by the EWO-8 results plotted in Fig. 6.10, which lie closer to the RT3 results than do the EWO-6 results. Further discussion of the accuracy of the VDOM and the role of quadrature sets in determining this accuracy is discussed throughout the remainder of this chapter, and is summarized in Section 6.10.
Figure 6.10 Results of Test 5 for $\phi = 0$ deg.
Figure 6.11 Results of Test 5 for $\phi = 90$ deg.
Figure 6.12 Results of Test 5: effect of VDOM quadrature set on Q Stokes parameter with $\phi = 0$ deg and V Stokes parameter with $\phi = 90$ deg.
Table 6.4 Comparison for Test 5 of numerical values generated by Evans (Evans and Stephens, 1991) and VDOM with $\phi = 0$ and 180 deg. The Stokes parameters have units of $W \cdot m^{-2} \cdot \mu m^{-1} \cdot sr^{-1}$.

### a) $\phi = 0$ deg.

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<th>I (VDOM)</th>
<th>Q (Evans)</th>
<th>Q (VDOM)</th>
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### b) $\phi = 180$ deg.

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<th>Q (VDOM)</th>
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Table 6.5 Comparison for Test 5 of numerical values generated by Evans (Evans and Stephens, 1991) and VDOM with $\phi = 90$ deg. The Stokes parameters have units of W-m$^{-2}$-µm$^{-1}$-sr$^{-1}$.

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<th>V (Evans)</th>
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Results of Test 5 for $\phi = 90$ deg are plotted in Fig. 6.11 as a function of $\mu$. The results again compare the sixteen-angle RT3 values versus the VDOM user-angle values generated using the EWO-6 and EWO-8 quadratures. The VDOM results compare favorably to the RT3 results, and again, it appears that the EWO-8 results give better agreement than those obtained using the EWO-6 quadrature. The V results for the VDOM tend to fall off the RT3 curve, but do give the correct trend. Because the magnitude of $V$ is extremely small relative to the other 3 Stokes parameters for this case, it is likely that the VDOM results for $V$ would be acceptable for most applications.

To examine the effect of the VDOM quadrature set on the results of this test, Fig. 6.12 displays plots of $Q$ at $\phi = 0$ deg and $V$ at $\phi = 90$ deg, both as functions of $\mu$ and the VDOM quadrature set. Also, the sixteen-angle RT3 results are shown in the plot. The two Stokes parameters in Fig. 6.12 are used because they exhibit the most obvious differences from the RT3 results shown in Figs. 6.10 and 6.11. The quadratures used are the EWO, LSO, and LSH (level symmetric hybrid) of orders 6 and 8. These quadrature sets are defined by Fiveland (1991). Though the plots are slightly cluttered, inspection reveals that, in general, the best agreement between the RT3 and VDOM results comes from the LSO quadrature set, and that, in particular, the LSO-8 set gives better agreement than the LSO-6 set.

Up to this point, validation of the VDOM has relied upon visual comparison of its results with other solutions. However, Tables 6.4 and 6.5 list numerical values of $I$, $Q$, $U$, and $V$ as generated by RT3 and the VDOM. Values are given for the Stokes parameters at the eight values of $\mu$ used by the RT3 eight-angle Gaussian quadrature scheme. The VDOM results are generated using the user-angle generation technique with the LSO-8 quadrature. Table 6.4 lists values of $I$ and $Q$ at $\phi = 0$ and 180 deg, and Table 6.5 lists all four Stokes parameters at $\phi = 90$ deg. Some summary statistics based on these values are presented in Section 6.10.

### 6.6 Test 6: Monodisperse Randomly-Oriented Prolate Spheroids

Test 6 is based on the "atmospheric model 1" of Wauben and Hovenier (1992) and Kuik et al. (1992).

#### 6.6.1 Description

The scenario used for Test 6 is the same as Test 5, and is depicted in Fig. 6.9. For this test, the atmosphere is comprised of a monodispersion of randomly-oriented prolate spheroids and is described more fully by the parameters listed in Table 6.2. The
The normalized Mueller matrix for the particle ensemble is computed using the generalized spherical function expansion coefficients listed in Table 6.6 in conjunction with the formulas given in Appendix 2.5. The phase matrix is obtained by multiplying the normalized Mueller matrix by the single scattering albedo.

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</table>
6.6.2 Results

Results of Test 6 for \( \phi = 90 \) deg are presented in Figs. 6.13 (I and Q) and 6.14 (U and V). Results are plotted for the four Stokes parameters at the user-angles of \( \mu \) listed previously, and are generated using LSO quadratures of orders 6, 8, 12, and 16, and the LSH-10 quadrature. The LSH-10 quadrature is used in place of the LSO-10 quadrature, which is not used because it contains negative weights. Also plotted are the results of Wauben and Hovenier (1992), which they obtained using an adding/doubling formulation of the VRTE, and verified using the \( F_N \) method. Two sets of results are shown per Stokes parameter for each VDOM solution: the first set does not apply the renormalization (labeled "normalized" in the figures) given by Eq. (5.24), and the second set does apply it (labeled "unnormalized"). This is investigated because, for this particular case, the normalization requirement given by Eq. (3.32) is in general not satisfied by any of the DOM quadrature sets applied to this problem. For example, in terms of the DOM, Eq. (3.32) requires that, for a given discrete outgoing direction \( i \)

\[
\frac{1}{4\pi} \sum_{j=1}^{8K} \Phi_{11,ij} w_j = 1
\]

(6.2)

In Table 6.7, the average and maximum values of Eq. (6.2) over all discrete outgoing directions are listed for each DOM quadrature set used in Test case 6. The values in the table indicate that, for this test case, the DOM quadrature sets gives values that are on average between about one and eighteen per cent too large.

Nevertheless, for Test 6, it appears that reasonably good accurate results are possible for I, Q, and U, as long as the renormalization of Eq. (5.24) is performed. For example, in Figs. 6.13 and 6.14, the "normalized" results for the I, Q, and U Stokes parameters are clearly clustered much closer to the Wauben and Hovenier results than are the "unnormalized" results. However, there are individual points for which this is not true: the LSO-16 values of Q and U at \( \mu = 0.09501 \), among others.
Figure 6.13 Results of Test 6 for $\phi = 90$ deg: effect of VDOM quadrature and phase matrix renormalization on I and Q Stokes parameters.
Figure 6.14 Results of Test 6 for $\phi = 90$ deg: effect of VDOM quadrature and phase matrix renormalization on $U$ and $V$ Stokes parameters.
Table 6.7 Phase function normalization values for Test 6.

<table>
<thead>
<tr>
<th>Quadrature</th>
<th>Average</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSO-6</td>
<td>1.180877</td>
<td>1.303464</td>
</tr>
<tr>
<td>LSO-8</td>
<td>1.097570</td>
<td>1.178167</td>
</tr>
<tr>
<td>LSH-10</td>
<td>1.065212</td>
<td>1.119104</td>
</tr>
<tr>
<td>LSO-12</td>
<td>1.011543</td>
<td>1.047502</td>
</tr>
<tr>
<td>LSO-16</td>
<td>1.065177</td>
<td>1.210801</td>
</tr>
</tbody>
</table>

Though the "normalized" VDOM results for I, Q, and U are in reasonable agreement with the values given by Wauben and Hovenier (1992), the same cannot be said for the results for V, which, for both the "normalized" and "unnormalized" cases, are scattered widely about the "Wauben" curve. However, a close inspection of Fig. 6.14 reveals that, with the exception of the values at $\mu = 0.28160$, the LSO-12 quadrature gives results that are in fairly good agreement with those of Wauben and Hovenier (1992), especially for the "normalized" case. Interestingly, as listed in Table 6.7, the LSO-12 set does the best job among the quadrature sets used for this Test of satisfying Eq. (6.2). Thus, it appears that a rule-of-thumb for achieving accurate results with the VDOM is to choose the quadrature that best satisfies the phase function normalization requirement. This suggestion is considered further in Section 6.10.

6.7 Test 7: Monodisperse Randomly-Oriented Oblate Spheroids

Test 7 is based on the "atmospheric model 2" of Wauben and Hovenier (1992) and Kuik et al. (1992).

6.7.1 Description

The scenario used for this test is as in the previous two tests, and is depicted in Fig. 6.9. The model atmosphere is composed of a monodispersion of randomly-oriented oblate spheroids, as described in Table 6.2. The phase matrix is obtained as in Section 6.6.1, except that the generalized spherical function expansion coefficients used for this case are those listed in Table 6.8.
Table 6.8 Generalized spherical function expansion coefficients of the phase matrix elements for Test 7. From Kuik et al. (1992).

<table>
<thead>
<tr>
<th>l</th>
<th>$\alpha_1^l$</th>
<th>$\alpha_2^l$</th>
<th>$\alpha_3^l$</th>
<th>$\alpha_4^l$</th>
<th>$\beta_1^l$</th>
<th>$\beta_2^l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.915207</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>1</td>
<td>2.104031</td>
<td>0.000000</td>
<td>0.000000</td>
<td>2.095727</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>2.095158</td>
<td>3.726079</td>
<td>3.615946</td>
<td>2.008624</td>
<td>0.116688</td>
<td>0.065456</td>
</tr>
<tr>
<td>3</td>
<td>1.414939</td>
<td>2.202868</td>
<td>2.240516</td>
<td>1.436545</td>
<td>0.209370</td>
<td>0.221658</td>
</tr>
<tr>
<td>4</td>
<td>0.703593</td>
<td>1.190694</td>
<td>1.139473</td>
<td>0.706244</td>
<td>0.227137</td>
<td>0.097752</td>
</tr>
<tr>
<td>5</td>
<td>0.235001</td>
<td>0.391203</td>
<td>0.365605</td>
<td>0.238475</td>
<td>0.144524</td>
<td>0.052458</td>
</tr>
<tr>
<td>6</td>
<td>0.064039</td>
<td>0.105556</td>
<td>0.082779</td>
<td>0.056448</td>
<td>0.052640</td>
<td>0.009239</td>
</tr>
<tr>
<td>7</td>
<td>0.012837</td>
<td>0.020484</td>
<td>0.013649</td>
<td>0.009703</td>
<td>0.012400</td>
<td>0.001411</td>
</tr>
<tr>
<td>8</td>
<td>0.002010</td>
<td>0.003097</td>
<td>0.001721</td>
<td>0.001267</td>
<td>0.002093</td>
<td>0.000133</td>
</tr>
<tr>
<td>9</td>
<td>0.000246</td>
<td>0.000366</td>
<td>0.000172</td>
<td>0.000130</td>
<td>0.000267</td>
<td>0.000011</td>
</tr>
<tr>
<td>10</td>
<td>0.000024</td>
<td>0.000035</td>
<td>0.000014</td>
<td>0.000011</td>
<td>0.000027</td>
<td>0.000001</td>
</tr>
<tr>
<td>11</td>
<td>0.000002</td>
<td>0.000003</td>
<td>0.000001</td>
<td>0.000001</td>
<td>0.000002</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
6.7.2 Results

Some results of Test 7 are presented in Fig. 6.15 and Tables 6.9 to 6.12. The VDOM results are generated using the LSO-8 quadrature, which satisfies Eq. (6.2) reasonably well. Values of the upwelling Stokes parameters at the top of the atmosphere are plotted with ten evenly spaced values of \( \mu \) from 0.1 to 1.0 with \( \phi = 0, 90 \) and 180 deg in Fig. 6.15. Also plotted in this figure are the results of Wauben and Hovenier (1992), who give results at the same values of \( \mu \) and \( \phi \), and in addition, give results for \( \mu = 0 \). In general, the VDOM results are in good agreement with the Wauben and Hovenier (1992) results. Very slight disagreement is evident for \( U \) with \( \mu \) between 0.2 and 0.7, and for \( V \) with \( \mu \) between 0.2 and 0.4, and again with \( \mu \) at 0.7 and 0.8.

Numerical values of upwelling \( I, Q, U, \) and \( V \) as generated by Wauben and Hovenier (1992) and the VDOM are given for \( \tau = 1 \) in Tables 6.9 (\( \phi = 0 \) and 180 deg) and 6.10 (\( \phi = 90 \) deg), and for \( \tau = 0.5 \) in Tables 6.11 (\( \phi = 0 \) and 180 deg) and 6.12 (\( \phi = 90 \) deg). Values of the Stokes parameters in each of these tables are given at ten values of \( \mu \) ranging from 0.1 to 1.0 in increments of 0.1, where the VDOM results are generated using the user-angle generation technique with the LSO-8 quadrature. On average, the VDOM results agree with the "Wauben" results to one or two significant digits. Wauben and Hovenier (1992) claim that their results are exact for the digits shown. Some summary statistics based on the values in these tables are presented in Section 6.10.

6.8 Test 8: Polydisperse Randomly-Oriented Prolate Spheroids

Test 8 is based on the "atmospheric model 3" of Wauben and Hovenier (1992) and Kuik et al. (1992).

6.8.1 Description

This test is depicted in Fig. 6.9, and considers an atmosphere consisting of a polydispersion of randomly-oriented prolate spheroids, as described in Table 6.2. This polydispersion is representative of an Earth volcanic aerosol. The phase matrix is obtained as in the previous two sections, but using the generalized spherical function expansion coefficients listed in Table 6.13.
Figure 6.15 Results of Test 7 for $\phi = 0$, 90 and 180 deg. $U$ and $V$ Stokes parameters are zero at 0 and 180 deg.
Table 6.9  Comparison of Test 7 numerical values generated by Wauben (Wauben and Hovenier, 1992) and VDOM with $\phi = 0$ and 180 deg, and $\tau = 1.0$. The Stokes parameters have units of W-m$^{-2}$-µm$^{-1}$-sr$^{-1}$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>I (Wauben)</th>
<th>I (VDOM)</th>
<th>Q (Wauben)</th>
<th>Q (VDOM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>9.2680E-1</td>
<td>9.1438E-1</td>
<td>-7.6404E-2</td>
<td>-7.5799E-2</td>
</tr>
<tr>
<td>0.2</td>
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<td>-6.3856E-2</td>
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</tr>
<tr>
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</tr>
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<td>4.5038E-1</td>
<td>-3.8770E-2</td>
<td>-3.8616E-2</td>
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<td>-2.8224E-2</td>
<td>-2.8199E-2</td>
</tr>
<tr>
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<td>2.5355E-1</td>
<td>-1.9778E-2</td>
<td>-1.9865E-2</td>
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<tr>
<td>0.7</td>
<td>1.8588E-1</td>
<td>1.8663E-1</td>
<td>-1.3513E-2</td>
<td>-1.3675E-2</td>
</tr>
<tr>
<td>0.8</td>
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<td>1.3473E-1</td>
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<td>-9.3637E-3</td>
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<td>0.9</td>
<td>9.3138E-2</td>
<td>9.3460E-2</td>
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<td>-6.3672E-3</td>
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<td>5.1553E-2</td>
<td>-2.6239E-3</td>
<td>-2.7784E-3</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$</th>
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<th>I (VDOM)</th>
<th>Q (Wauben)</th>
<th>Q (VDOM)</th>
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<td>1.9952E-3</td>
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</tr>
<tr>
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<td>5.0304E-2</td>
<td>-2.6239E-3</td>
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</table>
Table 6.10 Comparison of Test 7 numerical values generated by Wauben (Wauben and Hovenier, 1992) and VDOM with $\phi = 90$ deg and $\tau = 1.0$. The Stokes parameters have units of $W\cdot m^{-2}\cdot \mu m^{-1}\cdot sr^{-1}$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$I$ (Wauben)</th>
<th>$Q$ (Wauben)</th>
<th>$U$ (Wauben)</th>
<th>$V$ (Wauben)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-5.1492E-5</td>
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<td>2.6239E-3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$I$ (VDOM)</th>
<th>$Q$ (VDOM)</th>
<th>$U$ (VDOM)</th>
<th>$V$ (VDOM)</th>
</tr>
</thead>
<tbody>
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<td>2.6866E-3</td>
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</table>
Table 6.11 Comparison of Test 7 numerical values generated by Wauben (Wauben and Hovenier, 1992) and VDOM with $\phi = 0$ and 180 deg, and $\tau = 0.5$. The Stokes parameters have units of W-m$^{-2}$-um$^{-1}$-sr$^{-1}$.

### a) $\phi = 0$ deg.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$I$ (Wauben)</th>
<th>$I$ (VDOM)</th>
<th>$Q$ (Wauben)</th>
<th>$Q$ (VDOM)</th>
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<tbody>
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<td>-2.9805E-2</td>
<td>-3.0444E-2</td>
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<td>3.7499E-1</td>
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<td>-2.4194E-2</td>
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<td>-1.3395E-2</td>
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### b) $\phi = 180$ deg.

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<th>$Q$ (VDOM)</th>
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98
Table 6.12 Comparison of Test 7 numerical values generated by Wauben (Wauben and Hovenier, 1992) and VDOM with $\phi = 90$ deg and $\tau = 0.5$. The Stokes parameters have units of $\text{W-m}^{-2}\cdot\mu\text{m}^{-1}\cdot\text{sr}^{-1}$.

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6.8.2 Results

The values of the upwelling Stokes parameters at the top of the atmosphere for Test 8 as obtained by the VDOM and Wauben and Hovenier (1992) are plotted in Fig. 6.16 for $\phi = 0, 90,$ and 180 deg, and for the same values of $\mu$ as in Section 6.7. The VDOM results are computed using the LSO-8 quadrature and the user-angle generation technique. The VDOM results for I, Q, and U appear to be in excellent agreement with the Wauben and Hovenier (1992) results. The VDOM results for V have the correct trend, but seem to contain a fair amount of error, especially for $\mu < 0.6$. This is discussed further in Section 6.10.

6.9 Oriented Particles

For oriented particles, the elements of the phase matrix are functions of incoming and outgoing direction, and the elements of the extinction matrix and absorption vector are functions of outgoing direction. The VDOM is designed to accommodate oriented particles; however, to perform a polarized radiative transfer validation study for such particles, the phase matrix, extinction matrix, and absorption vector must be specified. Unfortunately, a code to compute these properties for input to the VDOM is not readily available, and thus, such a validation is left for a future study. It should be pointed out that the RT4 polarized radiative transfer code of Evans and Stephens (1995b) is capable of accommodating oriented particles in a 1-D system and, thus, it should be possible to compare VDOM and RT4 results for oriented particles given the appropriate input quantities.

6.10 Comments on Accuracy of Numerical Tests

In the preceding sections of this chapter, results of comparisons of the VDOM with other VRTE solution methods are presented in the form of figures containing plots and tables containing numerical values. The results of Test 6, presented in Section 6.6, suggests that the accuracy of the VDOM might be strongly influenced by how accurately a given DOM quadrature set satisfies Eq. (6.2). This notion is further examined in this section by presenting accuracy statistics of Tests 5 to 8 and comparing these statistics with phase function normalization values computed using Eq. (6.2).

In Table 6.14, average and maximum fractional differences and phase function normalization values for Tests 5 to 8 are listed. The fractional difference $I$ is defined as

$$f_I = \frac{|I_{\text{VDOM}} - I_{\text{Ref}}|}{|I_{\text{Ref,max}}|}$$

(6.3)
Figure 6.16 Results of Test 8 for φ = 0, 90 and 180 deg. U and V Stokes parameters are zero at 0 and 180 deg.
Table 6.14 Fractional differences and phase function normalization values for Tests 5 to 8. The fractional differences for Test 5 are computed using eight zenith angles and three azimuth angles, and for Tests 6 to 8 are computed using ten zenith angles and three azimuth angles. The phase function normalization values are computed using the LSO-8 quadrature.

a) Fractional differences for Tests 5 to 8.

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<td>0.0260</td>
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<tr>
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b) Phase function normalization values for Tests 5 to 8.

<table>
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</table>
where $I_{\text{vdom}}$ is the value of the VDOM result, $I_{\text{ref}}$ is the value of a reference result, and $I_{\text{ref,max}}$ is the value of the reference result that has the largest magnitude over the range of $\mu$ values for which results are obtained. Fractional differences for the other Stokes parameters are obtained by replacing $I$ with the appropriate quantity in Eq. (6.3). The fractional difference statistic is used by Evans and Stephens (1991), and so seems appropriate to also use here for consistency. For Table 6.14, all VDOM results are computed using the user-angle generation technique with the LSO-8 quadrature.

The fractional differences for Test 5 are computed using eight zenith angles and three azimuth angles (corresponding to the $\mu$ and $\phi$ values listed in Tables 6.4 and 6.5), and, for Tests 6 to 8, are computed using ten zenith angles and three azimuth angles ($\mu$ and $\phi$ of Tables 6.9 and 6.10). For Test 5, the reference results are the eight-angle Gaussian quadrature results of the RT3 model of Evans and Stephens (1991), and for Tests 6 to 8, the references results are those listed by Wauben and Hovenier (1992). The values listed in Table 6.14 are the average and maximum fractional differences computed over all zenith and azimuth angles.

For I, Q, and U, the VDOM average fractional differences are typically within about two per cent of the reference results, and the maximum fractional difference is usually about three percent, except for Test 6, where the maximum fractional difference for Q is about six percent. However, this is the case where the LSO-8 quadrature does not do very well at satisfying Eq. (6.2): on average, the normalization requirement is exceeded by almost ten percent.

The V parameter appears to always have the highest average and maximum fractional errors. This is perhaps because it is the Stokes parameter of the smallest magnitude for the cases presented: thus, its coupling with the other Stokes parameters could effectively magnify any errors present in the values of I, Q, and U. In terms of the average and maximum fractional errors for the Stokes parameters, the test cases can be ranked from smallest error to largest error in the order: Test 5, 7, 8, and 6. Interestingly, the average and maximum phase function normalization values can be ranked in the same order. This lends credence to the notion that the accuracy of the VDOM solution depends on how well the selected quadrature set integrates the phase function. For Test 6, in particular, none of the quadrature sets used in this study does a good job of accurately integrating the phase function. However, other DOM quadrature sets, such as those proposed by Koch et al. (1995), might perform better.
6.11 Tests of 2- and 3-D Polarized Radiative Transfer

Presently, no results are available for comparison with the VDOM for 2- and 3-D radiative transfer problems. A 3-D Monte Carlo code valid for thermal sources of radiation apparently has been developed (Kummerow, 1995), but the results have not been made publicly available. Thus, at the present time, there is no rigorous method of validating the VDOM for multi-dimensional problems. However, it is possible to conduct tests of the VDOM by performing radiative transfer computations on multi-dimensional systems that are homogeneous in one or two directions. For example, Test 5 can be recast as a 3-D problem by filling each cell of an \( NX \times NZ \) system with the 1-D atmosphere depicted in Fig. 6.9, and then dividing the resulting system into \( NY \) vertical layers. By applying periodic boundary conditions to the solution domain (described briefly in Section 5.1.4), the resulting solution should agree with the plane-parallel solution.

Test of 2- and 3-D polarized radiative transfer were performed as described in the preceding paragraph using the framework of Tests 3 and 5. The results are identical so long as exactly the same inputs are used for the tests (for example, the values of the finite-difference weighting factor \( \alpha \) defined by Eq. (5.13) must be identical for the 1-, 2-, and 3-D systems in order to produce identical output values). However, the number of iterations and amount of CPU time needed for higher dimensional problems increases relative to that needed for 1-D problems, as indicated by the results given in Table 6.15. These results are for Test 5, and are listed for 1- and 3-D polarized radiative transfer systems. The quadrature used is LSO-8, the number of vertical layers is \( NY = 10 \), \( \alpha = 0.55 \), and for the 3-D cases, the horizontal discretization is \( NX = NZ = 5 \). Numerical experiments indicate that the value of \( \alpha \) greatly affects the accuracy of the solution, and, for 3-D problems, greatly affects the computational time. As discussed in Section 5.1.2, \( \alpha = 0.5 \) gives a second order accurate solution, while \( \alpha = 1.0 \) produces a first order accurate solution. In the numerical tests, setting \( \alpha \) near 0.5 yielded solutions that agree better with the reference results than do results obtained using larger values of \( \alpha \), regardless of the dimensionality of the problem.

As for CPU time, the value of \( \alpha \) has little effect for 1-D problems. However, for 2- and 3-D problems, \( \alpha \) less than about 0.51 causes a dramatic increase in the number of iterations and the amount of CPU time needed for convergence. Thus, a rule-of-thumb for setting \( \alpha \) is: for 1-D problems, use \( \alpha = 0.5 \); and, for 2- and 3-D problems, use \( \alpha = 0.51 \). As indicated in Section 5.1.2, negative values of \( I \) are undesirable, but are possible for values of \( \alpha < 1.0 \). Thus, if a choice of \( \alpha \) yields \( I < 0 \), there are three possible
Table 6.15 Effect of dimensionality and source term linearization on Test 5. For all cases, the value of $\alpha$ is 0.55, and the number of vertical layers is $NY = 10$. For the 3-D cases, the horizontal discretization is $NX = NZ = 5$.

<table>
<thead>
<tr>
<th>Source Term Linearization</th>
<th>1-D iterations</th>
<th>CPU time*</th>
<th>1-D iterations</th>
<th>CPU time*</th>
</tr>
</thead>
<tbody>
<tr>
<td>off</td>
<td>48</td>
<td>~ 3.3 minutes</td>
<td>53</td>
<td>~ 18 minutes</td>
</tr>
<tr>
<td>on</td>
<td>45</td>
<td>~ 3.0 minutes</td>
<td>51</td>
<td>~ 17 minutes</td>
</tr>
</tbody>
</table>

*Configuration of virtual machine:
HP-9000 model 735/125 (160 MB RAM)
HP-9000 model 720 (48 MB RAM)
HP-9000 model 750 (64 MB RAM)
HP-9000 model 715/50 (64 MB RAM)

remedies: i) set the offending intensity equal to zero; ii) refine the grid spacing; or iii) increase $\alpha$. However, it must be remembered that negative values are only unwanted for I; zero and negative values are possible for the other Stokes parameters. An evaluation of the finite difference factor for multi-dimensional radiative transfer computations using the DOM has been performed by Chai et al. (1994), and perhaps some of their findings could be applied to the VDOM.

Finally, a few words are offered regarding the distributed computing method (PVM) used for the VDOM. As indicated in Table 6.15, the 1-D version of Test 5 consumes approximately three minutes of real time to achieve a solution (in a workstation environment). In contrast, the RT3 code of Evans and Stephens (1991) requires on the order of ten seconds to achieve a solution. For the 1-D VDOM, approximately ninety percent of the time is spent performing network communications. However, for the 3-D VDOM, only about fifty percent of the time is spent performing communications. The implication is that, as the number of computational grid cells increases, more of a benefit is gained by using the distributed computing methodology. Thus, the real strength of the VDOM is not for solving 1-D problems, but for solving 2- and especially 3-D problems.
6.12 Test of Source Term Linearization

The VDOM source term linearization, derived in Section 5.2.3, is applied to Test 5, and the results are included in Table 6.15. The results show that, for Test 5, application of the source term linearization to the 1-D problem saves three iterations (approximately twenty seconds of CPU time), and, for the 3-D problem, the technique saves two iterations (approximately a minute of CPU time). Though these do not appear significant, they are savings nonetheless. Because the source term linearization should never cause additional iterations to be taken (Chai et al. 1992), it should always be used for the VDOM.
CHAPTER VII
EXAMPLES OF NEW CAPABILITIES

In this chapter, the VDOM is applied to several example 1-D and 3-D problems. The first example problem considers a two-layer precipitating system with polydisperse spherical particles and compares 1-D versus 3-D polarized radiative transfer results. The second example problem considers the same precipitating system, but uses oblate spheroidal particles. The third problem considers Mie scattering of a collimated beam in a 3-D setting, and the final problem simulates polarized radiative transfer of a collimated beam through a furnace containing a polydispersion of prolate spheroids. Examples 1, 2, and 3 use the LSO-8 quadrature set, and Example 4 uses the LSO-6 quadrature set. For each Example, the phase function normalization requirement of Eq. (6.2) is satisfied using the specified quadrature set.

7.1 Example 1: Two-Layer Precipitating System with Polydisperse Spherical Particles

Example 1 is similar to Test 3 of Chapter VI, but the dimensions of the system and properties of the particles have been modified. Results for 1-D and 3-D polarized radiative transfer are presented.

7.1.1 Description of Problem

A schematic of the atmospheric system used in Example problems 1 and 2 is shown in Fig. 7.1, where the properties of the system are listed in Table 7.1. The system consists of a 2 km thick, two-layer precipitating cloud over a specular water surface. The complex refractive index of the surface is \( m = 3.724 - 2.212i \). The top half of the cloud contains randomly oriented spherical ice particles, and the bottom half contains randomly oriented spherical raindrops. Each layer also contains atmospheric gases that, for the radiative transfer model, are described in terms of an absorption coefficient \( k_{ag} \). The temperature profile within each layer varies linearly such that \( T = 280 \) K at the surface, \( T = 273 \) K at the ice/rain interface, and \( T = 267 \) K at \( y = 2 \) km.
Figure 7.1  Schematic for Examples 1 and 2.
Table 7.1 Properties of the two-layer atmosphere of Examples 1 and 2.

<table>
<thead>
<tr>
<th>Layer type</th>
<th>Spheres</th>
<th></th>
<th>Spheres</th>
<th></th>
<th>Oblate Spheroids</th>
<th></th>
<th>Oblate Spheroids</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ice</td>
<td>Rain</td>
<td>Ice</td>
<td>Rain</td>
<td>Ice</td>
<td>Rain</td>
<td>Ice</td>
<td>Rain</td>
</tr>
<tr>
<td>Height (km)</td>
<td>8.0</td>
<td>4.0</td>
<td>8.0</td>
<td>4.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rainrate (mm h(^{-1}))</td>
<td>3.0</td>
<td>2.0</td>
<td>3.0</td>
<td>2.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Refractive index</td>
<td>(1.7829,</td>
<td>(3.2781</td>
<td>(1.7829,</td>
<td>(3.2781</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m(r), m(i))</td>
<td>0.0034</td>
<td>1.8512</td>
<td>0.0034</td>
<td>1.8512</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(k_e) (km(^{-1}))</td>
<td>0.26061</td>
<td>0.51557</td>
<td>0.23609</td>
<td>0.51578</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(k_{ag}) (km(^{-1}))</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\omega)</td>
<td>0.98437</td>
<td>0.44137</td>
<td>0.98299</td>
<td>0.44135</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(r_{eff}) (mm)</td>
<td>0.4608</td>
<td>0.4232</td>
<td>0.4608</td>
<td>0.4232</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a / b</td>
<td>1.0</td>
<td>1.0</td>
<td>4.0</td>
<td>1.04</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Brightness temperature results are sought for 1- and 3-D radiative transfer at a frequency of 85.5 GHz. For the 3-D computations, the top and side view of the system is depicted in Fig. 7.2, where the shaded area in the figure contains the rain system of Fig. 7.1. The rainshaft is 24 by 24 km in the horizontal, and it is placed in the middle of an atmosphere with horizontal dimensions of 120 by 120 km. The horizontal size of the rainshaft is similar to the field-of-view of the 85.5 GHz channels for the SSM/I. The non-shaded areas in the figure are assumed to contain only atmospheric gases, with kag given in Table 7.1. The dimensions of the computational grid are $5 \times 10 \times 5$ (NX $\times$ NY $\times$ NZ). Periodic boundaries are assumed.

The radiative properties of the raining system are computed using the T-matrix method devised by Mishchenko (1991), where the precipitating particles within each layer are polydisperse, described by a gamma distribution

$$n(r) = N_v \frac{(r_{\text{eff}} v_{\text{eff}})^{-\alpha}}{\Gamma(\alpha)} r^{\alpha-1} \exp\left(-\frac{r}{r_{\text{eff}} v_{\text{eff}}} \right) \quad (7.1a)$$

In Eq. (7.1a), $N_v$ is the total number of particles per unit volume, $r_{\text{eff}}$ is the effective (average) radius of the polydispersion, $v_{\text{eff}}$ is the dimensionless effective variance of the distribution, $\Gamma()$ is the gamma function, and

$$\alpha = \frac{1 - 2 v_{\text{eff}}}{v_{\text{eff}}} \quad (7.1b)$$

Note the Marshall-Palmer DSD given by Eq. (4.19) is a special form of Eq. (7.1). According to Mishchenko and Travis (1994b), the scattering properties of distributions of spheres or spheroids are nearly independent of the DSD for a given effective radius and variance. In addition, they found that, to minimize the computational burden of single-scattering calculations, a power-law distribution

$$n(r) = \begin{cases} \frac{2 r_1^2 r_2^2}{r_2^2 - r_1^2} r^{-3} N_v & \text{for } r_1 \leq r \leq r_2 \\ 0 & \text{otherwise} \end{cases} \quad (7.2a)$$

should be used, where $r_1$ and $r_2$ are the minimum and maximum radius of the power-law distribution. The effective radius and variance of the power-law distribution are given by
Figure 7.2 Views of 3D raining system for Examples 1 and 2. The shaded area contains the rain system depicted in Fig. 7.1. Non-shaded areas contain only atmospheric gases. Periodic boundaries are assumed.
\[ r_{\text{eff}} = \frac{r_2 - r_1}{\ln \left( \frac{r_2}{r_1} \right)} \]  

(7.2b)

and

\[ v_{\text{eff}} = \frac{r_2 + r_1}{2} \frac{1}{r_2 - r_1} \ln \left( \frac{r_2}{r_1} \right) - 1 \]  

(7.2c)

Thus, for Example 1, the radiative properties of the ice and rain layers are computed by assuming a Marshall-Palmer rainrate, and then recasting Eq. (4.19) into the form of Eq. (7.2) using \( r_{\text{eff}} \) and \( v_{\text{eff}} \) as formal parameters. For the Marshall-Palmer distribution, \( v_{\text{eff}} = 1/3 \). The computed radiative properties are the extinction coefficient and single scattering albedo, listed in Table 7.1, and the phase matrix, generated using Eqs. (D.14) - (D.19) with the generalized spherical function expansion coefficients given in Table 7.2.

Table 7.2 Generalized spherical function expansion coefficients of the phase matrix elements for Example 1.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( \alpha_1^l )</th>
<th>( \alpha_2^l )</th>
<th>( \alpha_3^l )</th>
<th>( \alpha_4^l )</th>
<th>( \beta_1^l )</th>
<th>( \beta_2^l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.029790</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>1</td>
<td>0.052382</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>1.499265</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>2</td>
<td>0.500125</td>
<td>2.999109</td>
<td>0.097923</td>
<td>0.027687</td>
<td>1.222664</td>
<td>-0.053053</td>
</tr>
<tr>
<td>3</td>
<td>0.005116</td>
<td>0.017052</td>
<td>0.000411</td>
<td>0.000163</td>
<td>0.009332</td>
<td>-0.000319</td>
</tr>
<tr>
<td>4</td>
<td>0.000023</td>
<td>0.000056</td>
<td>0.000001</td>
<td>0.000000</td>
<td>0.000036</td>
<td>-0.000001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( l )</th>
<th>( \alpha_1^l )</th>
<th>( \alpha_2^l )</th>
<th>( \alpha_3^l )</th>
<th>( \alpha_4^l )</th>
<th>( \beta_1^l )</th>
<th>( \beta_2^l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.020989</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>1</td>
<td>0.041489</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>1.500141</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>2</td>
<td>0.500285</td>
<td>3.000165</td>
<td>0.074084</td>
<td>0.027170</td>
<td>1.224513</td>
<td>-0.000015</td>
</tr>
<tr>
<td>3</td>
<td>0.006671</td>
<td>0.022234</td>
<td>0.000386</td>
<td>0.000181</td>
<td>0.012177</td>
<td>0.000000</td>
</tr>
<tr>
<td>4</td>
<td>0.000037</td>
<td>0.000093</td>
<td>0.000001</td>
<td>0.000001</td>
<td>0.000059</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
7.1.2 Results

Upwelling brightness temperatures at the top of the atmosphere as a function of viewing angle for Example 1 are listed in Table 7.3 and plotted in Fig. 7.3. Results are presented for 1- and 3-D radiative transfer computations in terms of the horizontally and vertically polarized brightness temperatures. The U and V Stokes parameters for Example 1 are identically equal to zero due to the symmetry of the problem, and thus are not given. The LSO-8 quadrature set is used, and NY = 10 vertical layers for all cases. For the 1-D brightness temperatures, numerical results are obtained using the scalar DOM (Section 5.1.1), and the VDOM (Section 5.1.3). For the scalar DOM, polarization is accounted for through the surface boundary conditions, whereas for the VDOM, polarization is accounted for through the surface boundary conditions and particle scattering, extinction, and emission. In general, the 1-D DOM results are within a few degrees K of the 1-D VDOM results for a given polarization state, though for vertical polarization and \( \mu = 0.18127 \), the difference is nearly 7 K. A likely explanation for the similarity of the 1-D results is that radiation that is polarized by the specular surface becomes depolarized due to the spherical particles in the rain and ice layers, and, therefore, the scalar (unpolarized) DOM gives reasonable results.

The 3-D polarized radiative transfer results of Table 7.3b and Fig. 7.3 are, in general, quite different from the 1-D results. This is because, for 3-D radiative transfer, energy is transported laterally into horizontally adjacent columns. For Example 1, the rainshaft is surrounded by non-raining atmosphere, and thus, the brightness temperatures for the raining column are effectively smoothed out by the adjacent non-raining columns. Part c) of Table 7.3 lists the upwelling brightness temperatures for the non-raining columns. Comparison of parts a), b) and c) of Table 7.3 does not necessarily make it clear how the non-raining atmosphere adjacent to the rainshaft influences the upwelling brightness temperatures from the rainshaft.
Table 7.3 Upwelling brightness temperature results (K) for Example 1.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18127</td>
<td>153.91</td>
<td>149.98</td>
<td>160.65</td>
<td>153.44</td>
</tr>
<tr>
<td>0.57735</td>
<td>204.55</td>
<td>192.26</td>
<td>206.51</td>
<td>191.07</td>
</tr>
<tr>
<td>0.79612</td>
<td>206.82</td>
<td>196.01</td>
<td>206.84</td>
<td>196.04</td>
</tr>
<tr>
<td>0.96658</td>
<td>203.00</td>
<td>198.44</td>
<td>201.44</td>
<td>199.35</td>
</tr>
</tbody>
</table>

b) 3-D (VDOM).

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18127</td>
<td>191.31</td>
<td>143.40</td>
</tr>
<tr>
<td>0.57735</td>
<td>203.26</td>
<td>171.21</td>
</tr>
<tr>
<td>0.79612</td>
<td>198.82</td>
<td>182.76</td>
</tr>
<tr>
<td>0.96658</td>
<td>194.07</td>
<td>191.46</td>
</tr>
</tbody>
</table>

c) No rain.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18127</td>
<td>266.17</td>
<td>152.01</td>
</tr>
<tr>
<td>0.57735</td>
<td>222.01</td>
<td>136.52</td>
</tr>
<tr>
<td>0.79612</td>
<td>191.38</td>
<td>152.55</td>
</tr>
<tr>
<td>0.96658</td>
<td>171.76</td>
<td>165.78</td>
</tr>
</tbody>
</table>
Figure 7.3 Upwelling brightness temperature results for Example 1.
For example, it is reasonable to expect that the upwelling brightness temperature for a given column in a 3-D atmosphere is some weighted average of the brightness temperatures obtained using 1-D radiative transfer on several of the columns in the vicinity of the given column (in cloud-climate studies, this approach is called the "independent pixel approximation; see for example Cahalan et al., 1994). However, for polarized radiative transfer, care must be exercised in using this approach. For example, at a viewing angle of \( \mu = 0.18127 \), it seems reasonable to expect that the horizontally polarized brightness temperature for the 3-D raining atmospheres to lie between the 1-D raining and non-raining results, but this is not the case. This observation is explained by recalling that the horizontally and vertically polarized brightness temperatures are a measure of the polarization state as well as the magnitude of the intensity of the electromagnetic waves. Thus, to shed some additional light on the results of Example 1, the degree of polarization (Z) and linear depolarization ratio (LDR), defined by Eqs. (3.10) and (3.11), are computed and presented in Table 7.4. In all cases, Z and LDR for 3-D radiative transfer lie between the corresponding non-raining and 1-D raining Z and LDR. This means that the polarization state of the upwelling electromagnetic wave for 3-D cases may be interpreted as some average of the polarization states of non-raining and 1-D raining atmospheres. A similar argument applies to the magnitude of the intensity, given by the first Stokes parameter: it should lie between the values of I for the corresponding non-raining and 1-D raining atmospheres.

Finally, it should be clarified that the numbers listed in Table 7.3c are computed using the 3-D VDOM and are from the top of the atmospheric column "far" from the rainshaft (those corresponding to the corners of Fig. 7.2). Because these columns are "far" from the rainshaft, they are not influenced by 3-D effects, and polarization effects are only due to the underlying surface. Thus, the brightness temperature results for the non-raining atmosphere computed using the DOM and VDOM agree with the numbers in Table 7.3 to within a few tenths of a degree K.
Table 7.4 Degree of polarization ($Z$) and linear depolarization ratio (LDR) results for Example 1.

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$Z$</th>
<th>LDR</th>
<th>$Z$</th>
<th>LDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 1-D.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.18127</td>
<td>0.0131</td>
<td>0.9741</td>
<td>0.0233</td>
<td>0.9545</td>
</tr>
<tr>
<td></td>
<td>0.57735</td>
<td>0.0313</td>
<td>0.9393</td>
<td>0.0392</td>
<td>0.9245</td>
</tr>
<tr>
<td></td>
<td>0.79612</td>
<td>0.0271</td>
<td>0.9472</td>
<td>0.0271</td>
<td>0.9473</td>
</tr>
<tr>
<td></td>
<td>0.96658</td>
<td>0.0115</td>
<td>0.8773</td>
<td>0.0053</td>
<td>0.9895</td>
</tr>
<tr>
<td>b) 3-D (VDOM).</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.18127</td>
<td>0.1449</td>
<td>0.7469</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.57735</td>
<td>0.0865</td>
<td>0.8407</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.79612</td>
<td>0.0425</td>
<td>0.9184</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.96658</td>
<td>0.0068</td>
<td>0.9864</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) No rain.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.18127</td>
<td>0.2757</td>
<td>0.5678</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.57735</td>
<td>0.2412</td>
<td>0.6114</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.79612</td>
<td>0.1143</td>
<td>0.7949</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.96658</td>
<td>0.0179</td>
<td>0.9648</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
7.2 Example 2: Two-Layer Precipitating System with Polydisperse Oblate Spheroidal Particles

Example 2 considers the same raining system as in Example 1, but the precipitating particles are taken as oblate spheroids rather than spheres.

7.2.1 Description of Problem

The geometry for Example 2 is depicted in Fig. 7.1, where the precipitating ice and rain layers are assumed to consist of polydispersions of randomly oriented oblate spheroids. The spheroids are characterized in terms of a deformation parameter $a/b$, where $a$ and $b$ are the lengths of the semi-major and minor axes of the spheroids. For raindrops, the deformation parameter for the polydispersion is obtained using the relationship (Oguchi, 1983)

$$b/a = 1 - 0.091 \, r_{\text{eff}}$$ (7.3)

This relationship agrees well with observations for individual liquid hydrometeors of a given radius, but is applied here to the entire polydispersion using the effective radius for convenience. For ice particles, the deformation parameter is taken as $a/b = 4.0$, corresponding to ice plates and needles observed in the atmosphere (Pruppacher and Klett, 1980). It is believed that, due to magnetic and aerodynamical forces, ice particles tend to orient themselves with their major axis aligned perpendicular to their fall direction (Pruppacher and Klett, 1980). However, for this problem, random orientation is assumed for simplicity in computing the radiative properties.

For Example 2, the particle effective radii and variances, system rainrates and geometry, and temperature distribution are prescribed, and are the same as in Example 1. The power-law distribution is used to compute the radiative properties of the spheroids, where the radius needed in Eq. (7.2) is that of an equal-volume sphere. The resulting extinction coefficients and single scattering albedos for this problem are given in Table 7.1, and the generalized spherical function expansion coefficients for the phase matrix elements are listed in Table 7.5. A comparison of the extinction coefficients and single scattering albedo (Table 7.1) based on particle shape reveals only a slight difference for ice and virtually no difference for raindrops. However, no conclusions should be extrapolated to raining cases in general based on this problem. For example, larger differences might be present for higher rainrates, different particle shapes, different particle composition, and other frequencies.
Table 7.5 Generalized spherical function expansion coefficients of the phase matrix elements for Example 2.

### a) Oblate raindrops ($a/b = 1.04$).

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.029790</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>1</td>
<td>0.052382</td>
<td>0.000000</td>
<td>0.000000</td>
<td>1.499265</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.500125</td>
<td>2.999109</td>
<td>0.097923</td>
<td>0.027687</td>
<td>1.222664</td>
<td>-0.053053</td>
</tr>
<tr>
<td>3</td>
<td>0.005116</td>
<td>0.017052</td>
<td>0.000411</td>
<td>0.000163</td>
<td>0.009332</td>
<td>-0.000319</td>
</tr>
<tr>
<td>4</td>
<td>0.000023</td>
<td>0.000056</td>
<td>0.000001</td>
<td>0.000000</td>
<td>0.000036</td>
<td>-0.00000</td>
</tr>
</tbody>
</table>

### b) Oblate ice particles ($a/b = 4.0$).

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.019426</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>1</td>
<td>0.053085</td>
<td>0.000000</td>
<td>0.000000</td>
<td>1.345814</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.469396</td>
<td>2.814625</td>
<td>0.083828</td>
<td>0.037414</td>
<td>1.148201</td>
<td>0.000019</td>
</tr>
<tr>
<td>3</td>
<td>0.011181</td>
<td>0.037276</td>
<td>0.000862</td>
<td>0.000472</td>
<td>0.020407</td>
<td>0.000000</td>
</tr>
<tr>
<td>4</td>
<td>0.000130</td>
<td>0.000325</td>
<td>0.000006</td>
<td>0.000004</td>
<td>0.000206</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Plots of the normalized Mueller matrix elements (Section 3.5.3) are presented for rain in Fig. 7.4, and for ice in Fig. 7.5. Also shown in these figures are the normalized Mueller matrix elements for the spherical particles used in Example 1. For the raindrops, the curves in Fig. 7.4 for spherical particles lie on top of the curves for the slightly oblate particles, indicating that the redistribution of polarized radiant energy due to scattering is not, for this Example, influenced by the slight shape deviation of the raindrops from spheres. On the other hand, the normalized Mueller matrix elements for the elongated ice particles assumed for Example 2 (Fig. 7.5) do differ from their spherical counterparts used in Example 1. A more in-depth discussion of Figs. 7.4 and 7.5 is outside the scope of this analysis, but the interested reader may refer to van de Hulst (1981), Mishchenko (1992a), and Mishchenko and Travis (1994b) for an explanation of the meaning and importance of the normalized Mueller matrix elements.

7.2.2 Results

Results of upwelling brightness temperatures at the top of the atmosphere for Example 2 are listed in Table 7.6. As in Example 1, all results are generated using the LSO-8 quadrature and NY = 10 vertical layers. The results are listed in terms of vertically and horizontally polarized brightness temperatures for 1-D and 3-D radiative transfer, and are presented in the same format as that used in Section 7.1. The results are not plotted because they show the same trend as the results for spherical particles. However, the differences between the brightness temperature results of Examples 1 and 2 are plotted in Fig. 7.6. These results show that, for the given raining atmospheres, brightness temperature differences due to particle shape range from about 0.5 to 5.0 K for the various radiative transfer models. In all cases, the differences are positive, indicating that the spherical particle atmospheres yield larger brightness temperatures. Another feature of interest is that the brightness temperature differences usually decrease with increasing μ, except for the 3D VDOM case for vertical polarization. Finally, for the 1-D cases, the vertically polarized brightness temperature differences are larger for a given viewing angle than the horizontally polarized brightness temperature differences. The same is true for 3-D cases and viewing angles greater than 0.57735. For the 3-D VDOM case with μ = 0.18127, the polarized brightness temperature difference for horizontal polarization is larger than that for vertical polarization.
Figure 7.4 Normalized Mueller matrix elements for spherical rain particles of Example 1 (solid-lines) and oblate rain spheroids of Example 2 (dashed-lines).
Figure 7.5 Normalized Mueller matrix elements for spherical ice particles of Example 1 (solid-lines) and oblate ice spheroids of Example 2 (dashed-lines).
Table 7.6 Upwelling brightness temperature results (K) for Example 2.

<table>
<thead>
<tr>
<th></th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 1-D.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.18127</td>
<td>158.84</td>
<td>155.03</td>
<td>165.24</td>
<td>158.03</td>
</tr>
<tr>
<td>0.57735</td>
<td>207.59</td>
<td>194.81</td>
<td>209.51</td>
<td>193.62</td>
</tr>
<tr>
<td>0.79612</td>
<td>208.89</td>
<td>197.79</td>
<td>208.95</td>
<td>197.88</td>
</tr>
<tr>
<td>0.96658</td>
<td>204.47</td>
<td>199.91</td>
<td>203.01</td>
<td>200.87</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b) 3-D (VDOM).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.18127</td>
<td>191.97</td>
<td>147.76</td>
</tr>
<tr>
<td>0.57735</td>
<td>205.73</td>
<td>173.21</td>
</tr>
<tr>
<td>0.79612</td>
<td>200.70</td>
<td>184.45</td>
</tr>
<tr>
<td>0.96658</td>
<td>195.64</td>
<td>193.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$T_{B,v}$</th>
<th>$T_{B,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>c) No rain.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.18127</td>
<td>266.17</td>
<td>152.01</td>
</tr>
<tr>
<td>0.57735</td>
<td>222.01</td>
<td>136.52</td>
</tr>
<tr>
<td>0.79612</td>
<td>191.38</td>
<td>152.55</td>
</tr>
<tr>
<td>0.96658</td>
<td>171.76</td>
<td>165.78</td>
</tr>
</tbody>
</table>
Figure 7.6 Brightness temperature differences between spheroidal particle results of Examples 2 and spherical particle results of Examples 1.
In Table 7.7, Z and LDR for Example 2 are listed. Interestingly, in most cases, the Z and LDR for the spherical precipitation particles (Table 7.6) agree with the Z and LDR for the spheroidal precipitation particles (Table 7.7) to at least two decimal places. A possible explanation for this is that the spheroidal raindrops use $a/b = 1.04$, so that they are nearly spherical. Thus, for this problem, the depolarizing raindrops may be a larger influence in determining the upwelling polarized brightness temperatures than the oblate ice particles.

In concluding this section, the reader is reminded that the emphasis of Examples 1 and 2 is to demonstrate the 3-D VDOM, to compare its results to 1-D scalar and vector radiative transfer, and to argue that the results are reasonable. No generalization of the results should be applied to actual raining systems. For example, in the scenarios examined, only a single frequency has been used, at a single rainrate, over a particular surface, and in a highly idealized system. No consideration has been given to horizontally oriented ice particles, which are certainly more realistic than the case of randomly oriented ice crystals examined here. In addition, because of the symmetry of the system, the U and V Stokes parameters are zero. In a more general system, inhomogeneities would contribute to non-zero U and V Stokes parameters. However, these types of cases could be investigated using the present model, so long as the radiative properties of the system (phase matrix, extinction matrix, emission vector) are specified.

7.3 Example 3: Mie Scattering of Collimated Beam

Example 3 recasts Test 5 of Chapter VI into a 3-D setting.

7.3.1 Description of Problem

The geometry for Example 3 is depicted in Fig. 7.7 and represents an isolated cuboidal cloud of optical dimensions $\tau_x$, $\tau_y$, and $\tau_z$ over a diffuse surface with reflectance $\rho_0$ and $T = 0$ K. The cloud is taken to be internally homogeneous and cold ($T = 0$ K), and has transparent sides. An unpolarized collimated beam of intensity $I_c$ is incident on the cloud along the direction defined by the solar zenith angle $\theta_c$ and solar azimuthal angle $\phi_c$, and is taken to be the only source of energy for the system. The parameters for the system are the same as those for Test 5 of Chapter VI, and are given in Tables 6.2 and 6.3.
Table 7.7 Degree of polarization (Z) and linear depolarization ratio (LDR) results for Example 2.

<table>
<thead>
<tr>
<th>μ</th>
<th>Z</th>
<th>LDR</th>
<th>Z</th>
<th>LDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18127</td>
<td>0.0123</td>
<td>0.9757</td>
<td>0.0226</td>
<td>0.9558</td>
</tr>
<tr>
<td>0.57735</td>
<td>0.0321</td>
<td>0.9378</td>
<td>0.0398</td>
<td>0.9234</td>
</tr>
<tr>
<td>0.79612</td>
<td>0.0276</td>
<td>0.9463</td>
<td>0.0275</td>
<td>0.9465</td>
</tr>
<tr>
<td>0.96658</td>
<td>0.0114</td>
<td>0.9775</td>
<td>0.0054</td>
<td>0.9874</td>
</tr>
</tbody>
</table>

b) 3-D (VDOM).

<table>
<thead>
<tr>
<th>μ</th>
<th>Z</th>
<th>LDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18127</td>
<td>0.1316</td>
<td>0.7675</td>
</tr>
<tr>
<td>0.57735</td>
<td>0.0868</td>
<td>0.8403</td>
</tr>
<tr>
<td>0.79612</td>
<td>0.0426</td>
<td>0.9182</td>
</tr>
<tr>
<td>0.96658</td>
<td>0.0068</td>
<td>0.9864</td>
</tr>
</tbody>
</table>

c) No rain.

<table>
<thead>
<tr>
<th>μ</th>
<th>Z</th>
<th>LDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18127</td>
<td>0.2757</td>
<td>0.5678</td>
</tr>
<tr>
<td>0.57735</td>
<td>0.2412</td>
<td>0.6114</td>
</tr>
<tr>
<td>0.79612</td>
<td>0.1143</td>
<td>0.7949</td>
</tr>
<tr>
<td>0.96658</td>
<td>0.0179</td>
<td>0.9648</td>
</tr>
</tbody>
</table>
Figure 7.7 Finite cloud geometry for Example 3. The vertical walls and top are transparent, and the surface is diffuse with $\rho_0 = 0.1$ and $T = 0$ K.
7.3.2 Results

The four Stokes parameters are computed as a function of position and direction using the VDOM. The spatial discretization is $10 \times 10 \times 10$ and, for the directional discretization, the LSO-8 quadrature is used. Contours for the spectral upwelling flux (Section 3.7) as a function of optical path length $\tau_0$ are plotted in Fig. 7.8 for the I and Q Stokes parameters and in Fig. 7.9 for the U and V Stokes parameters. The spectral upwelling flux is used because it represents the average upwelling spectral energy over a hemisphere for the location under consideration.

The plot of the spectral upwelling flux for the I Stokes parameter in Fig. 7.8 is conspicuous because of the band of concentric rings that are shaded dark at the outermost rings, and become progressively lighter towards the innermost ring. In the plot, dark shades represent low intensities, and light shades represent high intensities, in a relative sense. The cuboidal cloud is illuminated uniformly from the top by the collimated beam, however, the transparent edges allow radiant energy to "leak" out the sides of the cloud. In addition, the surface beneath the cloud diffusely reflects a portion of energy incident upon it, so some of this energy is reflected out the cloud walls. Thus, the higher intensities are found in the innermost rings. The rings are not centered within the plot frame because the collimated beam illuminates the cloud from an oblique angle.

These results are in contrast to those for 1-D polarized radiative transfer, which yields a uniform upwelling flux because there are no horizontal "outlets" for energy to escape. For the subsequent discussion, it is useful to introduce the albedo, defined as the average upwelling flux over the surface under consideration divided by the total incoming energy, or

$$f_{q+} = \frac{\hat{q}^+}{\mu_c I_c} \quad (7.4)$$

where the hat over the $q$ represents an average over area of the upwelling flux for a given Stokes parameter. For 1-D polarized radiative transfer, the results of Test 5 in Chapter VI yield $f_{q+,I} = 0.5841$, compared to $f_{q+,I} = 0.4405$ for the 3-D results. In other words, for the 1-D system, about fifty-eight percent of the incoming energy is reflected back, compared to only about forty-four percent for the 3-D system, due to losses out the sides of the cloud.

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Figure 7.8 Spectral upwelling flux from cloud top for I and Q Stokes parameters. The collimated beam is incident from an azimuthal direction normal to the left hand side of the figure.
Figure 7.9 Spectral upwelling flux from cloud top for U and V Stokes parameters. The collimated beam is incident from an azimuthal direction normal to the left hand side of the figure.
Edge effects are also present for the spectral upwelling flux for the Q Stokes parameter results plotted in Fig. 7.8. For the plot, which resembles a wave "ripple" in a tank of water, light shades represent a relatively high value of $q_0^+$, and dark shades represent lower values. Recalling that Q represents the level of linear polarization, the plot indicates that the highest levels of linear polarization are concentrated near $(\tau_{0,x}, \tau_{0,z}) = (0.0, 0.5)$. Also, because the fluxes are all positive, the upwelling radiant energy must be predominately vertically polarized. In the $\tau_{0,z}$-direction, the spectral fluxes gradually diminish, and in the $\tau_{0,x}$-direction, they diminish more rapidly. One possible explanation is related to the $\tau_{0,x}$-direction being the azimuthal direction of travel of the incoming collimated beam. For example, the phase function ($P_{11}$) for this problem tends to scatter most incoming energy back or nearly back into the original direction of travel. Therefore, most of the energy of the incoming beam tends to continue in its original direction, and, upon interacting with the diffusely reflecting surface, becomes unpolarized. Thus, the energy represented at higher values of $\tau_{0,x}$ is possibly the result of incoming collimated energy that has been depolarized by the underlying surface. For the $\tau_{0,z}$-direction, the maximum level of polarization occurs at the center line because the incoming collimated energy is being scattered multiple times by polarizing particles; near the edges, fewer of these scattering events take place before the energy is scattered out of the domain.

This behavior is in contrast to that for the homogeneous 1-D cloud of Test 5 in Chapter VI. For that problem, the upwelling fluxes are homogeneous. In particular, the albedo for the Q Stokes parameter is $f_{q^+,Q} = 0.0786$, compared to $f_{q^+,Q} = 0.0290$ for the 3-D results. Thus, the edges effectively reduce the level of polarization that the incoming collimated beam undergoes by interacting with the cloud.

The results of the spectral upwelling flux computations for the U Stokes parameter are plotted in Fig. 7.9. Recall from Chapter III that U represents the level of linear polarization that is neither horizontal nor vertical (that is, it gives the plane of polarization for linearly polarized energy). For this figure, darker shades represent negative values of the spectral flux, and lighter shades represent positive values. The plot shows that the flux for U is not uniform for the 3-D cloud, as it is for the 1-D cloud. For example, for Test 5 of Chapter VI, the spectral upwelling flux for U is identically zero. However, for this problem, the spectral upwelling flux for U varies between $\pm 0.025$ W-m$^{-2}$-\mu m$^{-1}$, and the magnitude of $q_{U}^+$ increases with increasing $\tau_{0,x}$. In addition, the plot for the 3-D $q_{U}^+$ shows an interesting "anti-symmetry". That is, $\tau_{0,z} = 0.5$ is a line of symmetry for which values of $q_{U}^+$ on one side of the line are of equal magnitude but opposite sign of their counterparts across the symmetry line. On the symmetry line, $q_{U}^+ = 0$. Because of this
anti-symmetry, the albedo for the U Stokes parameter for the 3-D cloud is \( f_{q+U} = 0.0 \), identical to the results for the 1-D cloud.

The spectral upwelling flux results for the V Stokes parameter, plotted in Fig. 7.9, also exhibit a symmetry on either side of the \( \tau_{0,z} = 0.5 \) line. Recall that the V Stokes parameter represents the ellipticity, or handedness, of the radiant energy. In the plot, dark shades represent negative values of flux, and light shades represent positive values. Therefore, these results indicate that ellipticity changes across the symmetry line. Also, it appears that the magnitude of \( q^+ \) is nearly inversely proportional to distance from a corner. Similar to the results for the U Stokes parameter, the anti-symmetry of the spectral upwelling flux yields \( f_{q+,V} = 0.0 \) for the 3-D cloud, identical to the 1-D cloud. Finally, the values of \( V \) in the contour plot indicate that the ellipticity of the spectral upwelling flux changes at about \( \tau_{0,x} = 0.8 \). The significance of this is not understood, and requires further investigation. The apparent "non-symmetry" of the plotted fluxes in the vicinity of \((\tau_{0,x}, \tau_{0,z}) = (0.8, 0.5)\) is a feature of the plotting program, and is not an accurate representation of the values in the dataset, which give perfect anti-symmetry as described above.

To conclude this section, plots of upwelling Stokes parameters (as opposed to fluxes) are presented in Fig. 7.10 as a function of upwelling azimuthal angle \( \phi \) for an upwelling polar angle cosine of \( \mu = 0.57735 \), for both the 1-D results of Test 5 of Chapter VI, and the 3-D results of this section. The azimuthal angle in the plot uses the same reference frame as the collimated beam. The 3-D results are for \((\tau_{0,x}, \tau_{0,z}) = (0.5, 0.5)\), where it is presumed that edge effects are least prevalent, and thus, allowing for a more direct comparison with the 1-D results. To keep the discussion brief, only a few comments are offered on the comparison. For all the plots in Fig. 7.10, the trends for a given Stokes parameter are the same irrespective of the dimensionality of the problem. However, in general, the magnitudes differ depending on whether the 1-D or 3-D cloud is being examined. The 3-D results for I and Q at a given value of \( \phi \) are always less in magnitude than the corresponding 1-D results, suggesting that edge is affecting the intensity and level of polarization even at the center of the cuboidal cloud. Also, for the particular value of \( \mu \) being examined, Q takes on negative values at certain values of \( \phi \) (indicating a tendency towards horizontal polarization). For the U Stokes parameter, the 3-D results for a given azimuthal angle are almost always larger in magnitude than the 1-D results, though for V, the reverse seems to be true. Finally, for all four Stokes parameters, there is symmetry about \( \phi = 90 \) deg for both the 1- and 3-D results. For I and Q, it is a "pure" symmetry, and for U and V, an "anti"-symmetry.

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Figure 7.10 Stokes parameters for Example 3 at $\mu = 0.57735$. 
7.4 Example 4: Illumination of a Furnace by Collimated Radiation

Example 4 represents a furnace being illuminated by a collimated source (for example, a laser beam) that is unpolarized, and uniformly illuminates one side of the furnace.

7.4.1 Description of Problem

The geometry for Example 4 is depicted in 7.11, where an unpolarized collimated beam is incident normally upon one side of a furnace containing soot particles and combustion gases. The dimensions of the furnace are 0.5 by 0.5 m in the z- and y-directions, and 1.0 m in the x-direction, which is the direction of propagation of the collimated beam. The furnace walls are each at a temperature of $T_w = 1000 \text{ K}$. The walls at $x = 0.0$ and $1.0 \text{ m}$ are transparent (hence are non-emitting, and the remaining walls are black. The energy from the collimated beam is $I_c = 100 \text{ W-m}^{-2}\text{-}\mu \text{m}^{-1}\text{-sr}^{-1}$ and concentrated at a wavelength of $0.6328 \mu \text{m}$. An idealized sensor capable of measuring polarized radiant energy quantities at this wavelength is placed facing the furnace, just beyond the furnace wall at $x = 1.0 \text{ m}$. It is assumed that the space between the furnace and sensor is radiatively non-participating.

![Furnace Geometry Diagram](image)

Figure 7.11 Furnace geometry for Example 4. The top, bottom, front, and back walls are black, and the left and right walls are transparent. $T_g$ is the internal gas and particle temperature, $T_w$ is the wall temperature.

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The particles and gases are at a uniform temperature of $T_g = 1400$ K, and are described in terms of radiative properties by a phase matrix, extinction matrix, single scattering albedo, and gaseous absorption coefficient. For this problem, two shapes of particles are considered: spheres, and prolate spheroids. Because these particle shapes are axisymmetric, the extinction matrix is diagonal, and thus can be described by a single extinction coefficient $k_e$. The particles are polydisperse and described by a gamma size distribution using the parameters $r_{\text{eff}}$ and $v_{\text{eff}}$. As in Section 7.1.1, the shape distribution is transformed to a power-law distribution to efficiently compute the radiative properties. Values of the various particle parameters for Example 4 are listed in Table 7.8, and the generalized spherical function expansion coefficients for the phase matrix elements are given in Table 7.9. It should be pointed out that the total number of particles for the sphere and spheroid cases is chosen such that $k_e$ for each case is 0.998 m$^{-1}$. In addition, the gaseous absorption coefficient for each case is 0.002 m$^{-1}$. Thus, the optical thicknesses for each case are 1.0 in the x-direction, and 0.5 in the z- and y-directions.

7.4.2 Results

Results for this problem are generated for the spherical and prolate spheroidal particle cases using the VDOM using the LSO-6 quadrature set and a grid discretization of $(NX \times NY \times NZ) = (10 \times 5 \times 5)$. The first sets of results are radiative fluxes observed by the sensor for each particle type, and are presented for the modified Stokes parameter $I_v$ in Fig. 7.12, and for the modified Stokes parameter $I_h$ in Fig. 7.13. Due to the symmetry of the problem, and because the collimated beam is incident normal to the wall, the fluxes for the U and V Stokes parameters are zero. For the results plotted in Figs. 7.12 and 7.13, the symmetry inherit in the configuration of the problem manifests itself as a series of concentric rings for the fluxes for both modified Stokes parameters.

The spectral fluxes for $I_v$, shown in Fig. 7.12, range from 0.02383 W-m$^{-2}$-µm$^{-1}$ emanating from the corners of the furnace to 0.03354 W-m$^{-2}$-µm$^{-1}$ from the center of the furnace for the spherical particles, and, for the spheroidal particles, range from 0.02263 to 0.03170 W-m$^{-2}$-µm$^{-1}$ from the corner and center, respectively. The average spectral flux for $I_v$ reaching the sensor is 0.02852 W-m$^{-2}$-µm$^{-1}$ for the spheres, and 0.027018 for the spheroids. These numbers, along with the overall darker level of shading in Fig. 7.12 for the spheroidal particle case, indicate that, in general, the prolate spheroidal particles reduce the level of vertically polarized radiation reaching the sensor compared to that for spherical particles.
Table 7.8 Parameters for Example 4.

<table>
<thead>
<tr>
<th></th>
<th>Sphere</th>
<th>Prolate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{\text{eff}}$ ($\mu$m)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$v_{\text{eff}}$</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>Refractive index</td>
<td>(1.79,</td>
<td>(1.79,</td>
</tr>
<tr>
<td>($m_r$, $m_i$)</td>
<td>0.79)*</td>
<td>0.79)*</td>
</tr>
<tr>
<td>$k_c$ ($m^{-1}$)</td>
<td>0.998</td>
<td>0.998</td>
</tr>
<tr>
<td>$k_{ag}$ ($m^{-1}$)</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.32504</td>
<td>0.305585</td>
</tr>
<tr>
<td>$a/b$</td>
<td>1.0</td>
<td>4.0</td>
</tr>
</tbody>
</table>

*from Modest (1993)
Table 7.9 Generalized spherical function expansion coefficients of the phase matrix elements for Example 4.

a) Spherical soot.

<table>
<thead>
<tr>
<th>j</th>
<th>$\alpha_1^l$</th>
<th>$\alpha_2^l$</th>
<th>$\alpha_3^l$</th>
<th>$\alpha_4^l$</th>
<th>$\beta_1^l$</th>
<th>$\beta_2^l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.026723</td>
<td>0.000000</td>
<td>0.000000</td>
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<tr>
<td>1</td>
<td>0.054152</td>
<td>0.000000</td>
<td>0.000000</td>
<td>1.500244</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.500505</td>
<td>3.000287</td>
<td>0.095801</td>
<td>0.036808</td>
<td>1.224190</td>
<td>-0.021498</td>
</tr>
<tr>
<td>3</td>
<td>0.009381</td>
<td>0.031262</td>
<td>0.000686</td>
<td>0.000336</td>
<td>0.017118</td>
<td>-0.000238</td>
</tr>
<tr>
<td>4</td>
<td>0.000074</td>
<td>0.000186</td>
<td>0.000003</td>
<td>0.000002</td>
<td>0.000118</td>
<td>-0.000001</td>
</tr>
</tbody>
</table>

b) Prolate soot ($a / b = 4.0$).

<table>
<thead>
<tr>
<th>j</th>
<th>$\alpha_1^l$</th>
<th>$\alpha_2^l$</th>
<th>$\alpha_3^l$</th>
<th>$\alpha_4^l$</th>
<th>$\beta_1^l$</th>
<th>$\beta_2^l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0.000000</td>
<td>0.000000</td>
<td>0.026639</td>
<td>0.000000</td>
<td>0.000000</td>
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<td>0.000000</td>
<td>1.237539</td>
<td>0.000000</td>
<td>0.000000</td>
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<tr>
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<td>0.450472</td>
<td>2.686707</td>
<td>0.108056</td>
<td>0.051291</td>
<td>1.097601</td>
<td>-0.005422</td>
</tr>
<tr>
<td>3</td>
<td>0.015833</td>
<td>0.052648</td>
<td>0.001939</td>
<td>0.001149</td>
<td>0.028837</td>
<td>-0.000095</td>
</tr>
<tr>
<td>4</td>
<td>0.000343</td>
<td>0.000857</td>
<td>0.000026</td>
<td>0.000017</td>
<td>0.000542</td>
<td>-0.000002</td>
</tr>
<tr>
<td>5</td>
<td>0.000005</td>
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<td>0.000000</td>
<td>0.000000</td>
<td>0.000007</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Figure 7.12 Spectral outgoing flux from furnace for modified Stokes parameter $I_v$. Top: spherical particles. Bottom: prolate spheroidal particles.
Figure 7.13  Spectral outgoing flux from furnace for modified Stokes parameter $I_h$. Top: spherical particles. Bottom: prolate spheroidal particles.
Similar observations are made for the spectral fluxes for $I_h$, plotted in Fig. 7.12. For the spherical particles, the fluxes measured by the sensor range from $0.04178 \text{ W-m}^{-2} \mu \text{m}^{-1}$ at the furnace corners to $0.05663 \text{ W-m}^{-2} \mu \text{m}^{-1}$ at the furnace center. The spheroidal particle fluxes range from $0.03752 \text{ W-m}^{-2} \mu \text{m}^{-1}$ at the corners to $0.05085 \text{ W-m}^{-2} \mu \text{m}^{-1}$ at the center. Thus, the level of horizontally polarized radiation reaching the sensor is higher for the spherical particle case than for the spheroidal particle case.

These findings are somewhat surprising because they indicate that, overall, the signal received by the sensor detects a higher level of polarization for the spherical particles than for the prolate spheroids. In fact, the average degree of polarization received by the sensor is about $Z = 26.4$ percent for the spheres compared to $Z = 23.9$ percent for the spheroids. This is surprising because, in general, non-sphericity contributes to increasing the degree of polarization. However, a possible explanation for the lower average value of $Z$ received by the sensor for the spheroids is that the single scattering albedo for the prolate particles is less than that for the polydispersion of spheres. In other words, relatively more energy is scattered by the spherical particles and, for this problem, scattering is the only contributor to polarization of the signal.

As a final check of the 3-D VDOM, Example 4 is run without the incident collimated beam, and the furnace is taken as an isothermal enclosure with $T_w = T_g = 1000$ K. The remaining parameters are as previously specified, and the sphere and spheroid cases are considered. For both cases, the computed radiation field is unpolarized. This is expected, because radiation inside an isothermal enclosure is blackbody radiation (Brewster, 1992), and thus is isotropic and unpolarized, regardless of the contents of the enclosure or the optical characteristics of the wall surfaces.

### 7.5 Summary

In this chapter, four example problems are presented. All of the problems consider polarized radiative transfer in 3-D systems. The first two problems consider only thermal sources of radiant energy in microwave atmospheric remote sensing scenarios. The results compare 1-D versus 3-D radiative transfer results using both the scalar (unpolarized) and vector (polarized) formulations of the radiative transfer equation. The third problem considers only a direct (collimated) source of radiant energy, and results are compared for 1- and 3-D polarized radiative transfer. Finally, the fourth problem considers both thermal and direct sources of radiation, and examines the effect of particle shape on polarized radiant fluxes leaving an idealized 3-D furnace. Because it does not appear that other 3-D polarized radiative transfer models exist, the 3-D results presented here can not be validated at this time. However, an attempt has been made to show that the results are reasonable.
CHAPTER VIII

SUMMARY AND RECOMMENDATIONS

In this chapter, the major developments of the study are summarized, and conclusions are presented. In addition, recommendations for future work are given.

8.1 Summary

The primary purpose of this research is to develop a multi-dimensional polarized radiative transfer model based on the discrete-ordinates method, and to validate this model against existing results where possible. A secondary aspect of the research is to apply the model to multi-dimensional systems, such as three-dimensional precipitating atmospheres, and to demonstrate that the model produces feasible results, and to show how the results differ from polarized and unpolarized plane-parallel results.

The underlying theory for radiative transfer and polarization is presented in Chapters III and IV. Of major importance is the definition of the vector of the four Stokes parameters, I, Q, U, and V, that describe the power, level of linear polarization, plane of polarization, and ellipticity of an electromagnetic wave. These quantities are convenient because they can be measured by constructing appropriate optical devices. The vector radiative transfer equation (VRT) describes how the Stokes vector changes as energy propagates through an absorbing, emitting, and anisotropically scattering medium. Many of the important quantities used throughout the text are defined in Chapter III, and the radiative properties that are needed for the VRT are discussed in Chapter IV.

The vector discrete ordinates method (VDOM) is formulated in Chapter V, and the general VDOM solution procedure is discussed. It is shown that the VDOM expressions for each Stokes parameter are of the same form as the DOM expression used to solve the scalar radiative transfer equation. In addition, computational aspects of the VDOM are described, including the VDOM solution procedure using a distributed computing approach. Details on inclusion of a collimated source term, implementation of a VDOM source term linearization technique, boundary condition formulation, and evaluation of output intensities in user-specified directions are also provided.

Extensive validation of the model is performed in Chapter VI: the model is applied to eight test cases to confirm that it functions properly for plane-parallel polarized radiative transfer in situations involving both thermal and collimated sources of radiation for emitting, absorbing, and anisotropically scattering media containing randomly oriented axisymmetric particles. However, it seems that the accuracy of the results are dependent on
the particular quadrature set selected for a given problem. For example, the VDOM solution for the Stokes vector can only be assured to be accurate when the chosen quadrature set satisfies the normalization condition for the phase function. The model has not been validated for 2- and 3-D problems, due to the lack of published results for these types of geometries. However, tests of the VDOM on multi-dimensional systems that are homogeneous in two directions show that the model gives results that are consistent with plane-parallel polarized radiative transfer.

In Chapter VII, the model is applied to several example 1-D and 3-D problems. The problems illustrate the importance of accounting for edge effects in polarized radiative transfer. Two example cases consider thermal sources of radiation for 1-D and 3-D microwave radiative transfer for precipitating atmospheres containing spherical and aspherical particles. A third example case considers 3-D infrared radiative transfer for a collimated source irradiating a cuboidal cloud. A final example case considers both thermal and collimated sources. Though the 3-D cases cannot be validated at the present time due to a lack of published data, the VDOM results appear to be feasible.

Because the model has been successfully developed, implemented, and tested, the complete set of objectives outlined in Chapter I has been satisfied. As far as being a unique contribution to the field of radiative transfer, the VDOM is, to the best of the author's knowledge, the first polarized radiative transfer solver in existence capable of solving 3-D problems in scattering media that included both thermal and collimated sources. In addition, the development of the VDOM represents an extension of a commonly used radiative transfer method (the scalar discrete ordinates method) to solve polarized radiative transfer problems.

8.2 Recommendations

Though the VDOM has been successfully developed, tested, and implemented, there are several possible directions that one could go in its continued development. For example, though the model is capable of performing polarized radiative transfer computations for multi-dimensional systems containing oriented (as opposed to randomly-oriented) particles, this capability has not been tested.

Other recommendations for future work include applying the model to more complicated and realistic inhomogeneous structures than those used in this research. For example, it was mentioned at the outset of this work that polarization signatures have been observed in microwave remote sensing studies of precipitating systems, and that these signatures are believed to be due to the presence of oriented ice particles. Though the example cases presented here model microwave radiative transfer through precipitating
systems containing ice particles, they are highly idealized scenarios and consider only randomly oriented particles at relatively low rainrates. Thus, in the future, more realistic drop size, shape, and orientation distributions should be considered within the VDOM framework, and the results should be compared with observations.

Another area in which the model can be improved is computational efficiency. The model is designed to run on a network of heterogeneous computers, including workstations, supercomputers, and/or massively parallel machines. However, it has not been optimized to run on any of these platforms. Perhaps the most "cumbersome" aspect of the model is that it appears to be slowed down by the communication of partial results between computer processors. No attempt has been made at this point to improve the communication efficiency, and doing so would certainly make the model more attractive.

Several limitations of the model are listed in Section 1.3.3. Overcoming any one of these items would certainly improve the model. As other multi-dimensional polarized radiative transfer solvers become available, their results should be compared with the VDOM in order to ascertain the accuracy of the models. Finally, comparison of model results with experimental observations is desirable.
REFERENCES


APPENDIX A.

ATMOSPHERIC GASEOUS ABSORPTION AND EMISSION

A.1 Introduction

Ulaby et al. (1981) provide useful formulas for computing the absorption coefficients of water vapor and oxygen in the spectral region 1-300 GHz. These formulas are presented here for convenience.

A.2 Water Vapor

In the microwave region, water vapor has absorption lines at 22.235 and 183.31 GHz, along with numerous other lines at frequencies just above this region. Ulaby et al. (1981) give a formula for computing the absorption coefficient (dB/km) of water vapor at a frequency \( f \) (GHz) as

\[
\tilde{k}_{a,H_2O}(f) = k_{a,H_2O}(f) + \Delta k_{a,H_2O}(f)
\]  

(A.1)

where the tilde denotes that the absorption coefficient consists of a theoretically derived term and an empirical correction factor. The first term on the right hand side of Eq. (A.1) is the theoretically derived portion of the water vapor coefficient (see Section 4.2), calculated from

\[
k_{a,H_2O}(f) = 2 f^2 D_v \left( \frac{300}{T} \right)^{5/2} \sum_{i=1}^{10} A_i e^{-\frac{\gamma_i}{f}} \left[ \frac{\gamma_i}{(f_i^2 - f^2)^2 + 4 f^2 \gamma_i^2} \right]
\]

(A.2)

where \( D_v \) is the atmospheric water vapor density in grams per cubic meter, and \( T \) is the atmospheric temperature (K). The linewidth parameter (GHz) is given by

\[
\gamma_i = \gamma_0 \left( \frac{P}{1013} \right) \left( \frac{300}{T} \right)^x \left[ 1 + 10^{-2} a_i \frac{D_v T}{P} \right]
\]

(A.3)
where $P$ is the atmospheric pressure (mbar). The second term on the right hand side of Eq. (A.1) is the empirical correction factor, given by

$$\Delta k_{a,H_2O}(f) = 4.69 \times 10^{-6} \; D_v \left( \frac{300}{T} \right)^{2.1} \left( \frac{P}{1000} \right)^2 f^2$$  \hspace{1cm} (A.4)

The values of the line parameters $A_i$, $e_i$, $f_i$, $\gamma_{i0}$, $a_i$, and $x$ are listed in Table A.1 for $i = 1$ to 10.

### A.3 Oxygen

In the microwave portion of the spectrum, there are a large number of absorption lines centered near 60 GHz, and an additional line at 118.75 GHz. In the spectral region from 1-300 GHz, for an atmospheric oxygen concentration of 0.21 by volume, the oxygen absorption coefficient in air (dB/km) for a frequency $f$ (GHz) is given by (Ulaby, et al., 1981)

$$k_{a,O_2}(f) = 1.61 \times 10^{-2} \; f^2 \left( \frac{P}{1013} \right) \left( \frac{300}{T} \right)^2 \tilde{F}$$  \hspace{1cm} (A.5)

where $P$ is atmospheric pressure (mbar), and $T$ is atmospheric temperature (K). The line-shape function $\tilde{F}$ is calculated from the expression

$$\tilde{F} = \frac{0.7 \gamma_b}{f^2 + \gamma_b^2} + \sum_{N=1 \; \text{odd}}^{39} \Phi_N \left[ g_{N+}(f) + g_{N+}(-f) + g_{N-}(f) + g_{N-}(-f) \right]$$  \hspace{1cm} (A.6)

where

$$g_{N\pm}(f) = \frac{\gamma_N d_{N\pm}^2 + P \; (f - f_{N\pm}) \; Y_{N\pm}}{(f - f_{N\pm})^2 + \gamma_N^2}$$  \hspace{1cm} (A.7)

and
Table A.1 Line parameters of the ten lowest water vapor transitions  
(Ulaby et al., 1981).

<table>
<thead>
<tr>
<th>i</th>
<th>$i_i^h$ GHz</th>
<th>$e_i$ K</th>
<th>$A_i$</th>
<th>$\gamma_0$ GHz</th>
<th>$a_i$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22.23515</td>
<td>644</td>
<td>1.0</td>
<td>2.85</td>
<td>1.75</td>
<td>0.626</td>
</tr>
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<td>2</td>
<td>183.31012</td>
<td>196</td>
<td>41.9</td>
<td>2.68</td>
<td>2.03</td>
<td>0.649</td>
</tr>
<tr>
<td>3</td>
<td>(323.)</td>
<td>1850</td>
<td>334.4</td>
<td>2.30</td>
<td>1.95</td>
<td>0.420</td>
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<td>454</td>
<td>115.7</td>
<td>3.03</td>
<td>1.85</td>
<td>0.619</td>
</tr>
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<td>5</td>
<td>380.1968</td>
<td>306</td>
<td>651.8</td>
<td>3.19</td>
<td>1.82</td>
<td>0.630</td>
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<tr>
<td>6</td>
<td>(390.)</td>
<td>2199</td>
<td>127.0</td>
<td>2.11</td>
<td>2.03</td>
<td>0.330</td>
</tr>
<tr>
<td>7</td>
<td>(436.)</td>
<td>1507</td>
<td>191.4</td>
<td>1.50</td>
<td>1.97</td>
<td>0.290</td>
</tr>
<tr>
<td>8</td>
<td>(438.)</td>
<td>1070</td>
<td>697.6</td>
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<td>0.360</td>
</tr>
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<td>9</td>
<td>(442.)</td>
<td>1507</td>
<td>590.2</td>
<td>1.51</td>
<td>2.02</td>
<td>0.332</td>
</tr>
<tr>
<td>10</td>
<td>448.0008</td>
<td>412</td>
<td>973.1</td>
<td>2.47</td>
<td>2.19</td>
<td>0.510</td>
</tr>
</tbody>
</table>
\[ \Phi_N = 4.6 \times 10^{-3} \left( \frac{300}{T} \right) (2N + 1) \exp \left[ -6.89 \times 10^{-3} N (N + 1) \left( \frac{300}{T} \right) \right] \]  

(A.8)

The resonant and nonresonant linewidth parameters (GHz), respectively, are given by

\[ \gamma_N = 1.18 \left( \frac{P}{1013} \right) \left( \frac{300}{T} \right)^{0.85} \]  

(A.9a)

and

\[ \gamma_b = 0.49 \left( \frac{P}{1013} \right) \left( \frac{300}{T} \right)^{0.89} \]  

(A.9b)

The amplitudes of the resonant frequencies \( f_{N^+} \) and \( f_{N^-} \) are given by

\[ d_{N^+} = \left[ \frac{N (2N + 3)}{(N + 1)(2N + 1)} \right]^{1/2} \]  

(A.10a)

and

\[ d_{N^-} = \left[ \frac{(N + 1)(2N - 1)}{N (2N + 1)} \right]^{1/2} \]  

(A.10b)

The values of the resonant frequencies \( f_{N^+} \) and \( f_{N^-} \), and the interference parameters \( Y_{N^+} \) and \( Y_{N^-} \), are given in Table A.2 for \( N = 1 \) to 39.
Table A.2 Frequencies and interference coefficients (Ulaby et al., 1981).

<table>
<thead>
<tr>
<th>N</th>
<th>$f^+_N$</th>
<th>$f^-_N$</th>
<th>$Y^+_N$</th>
<th>$Y^-_N$</th>
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<tr>
<td>1</td>
<td>56.2648</td>
<td>118.7503</td>
<td>4.51E-4</td>
<td>-2.14E-5</td>
</tr>
<tr>
<td>3</td>
<td>58.4466</td>
<td>62.4863</td>
<td>4.94E-4</td>
<td>-3.78E-4</td>
</tr>
<tr>
<td>5</td>
<td>59.5910</td>
<td>60.3061</td>
<td>3.52E-4</td>
<td>-3.92E-4</td>
</tr>
<tr>
<td>7</td>
<td>60.4348</td>
<td>59.1642</td>
<td>1.86E-4</td>
<td>-2.68E-4</td>
</tr>
<tr>
<td>9</td>
<td>61.1506</td>
<td>58.3239</td>
<td>3.30E-5</td>
<td>-1.13E-4</td>
</tr>
<tr>
<td>11</td>
<td>61.8002</td>
<td>57.6125</td>
<td>-1.03E-4</td>
<td>3.44E-5</td>
</tr>
<tr>
<td>13</td>
<td>62.4112</td>
<td>56.9682</td>
<td>-2.23E-4</td>
<td>1.65E-4</td>
</tr>
<tr>
<td>15</td>
<td>62.9980</td>
<td>56.3634</td>
<td>-3.32E-4</td>
<td>2.84E-4</td>
</tr>
<tr>
<td>17</td>
<td>63.5685</td>
<td>55.7837</td>
<td>-4.32E-4</td>
<td>3.91E-4</td>
</tr>
<tr>
<td>19</td>
<td>64.1278</td>
<td>55.2214</td>
<td>-5.26E-4</td>
<td>4.93E-4</td>
</tr>
<tr>
<td>21</td>
<td>64.6789</td>
<td>54.6711</td>
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APPENDIX B.
MUELLER AND SCATTERING AMPLITUDE MATRICES

B.1 Introduction

The relationship between the incoming \((E_{vi}, E_{hi})\) and scattered \((E_{vs}, E_{hs})\) electromagnetic fields for a wave incident on a particle of arbitrary shape and size is given by the far-field expression (Bohren and Huffman, 1983)

\[
\begin{pmatrix}
E_{vs} \\
E_{hs}
\end{pmatrix} = \frac{\exp[ik(r-y)]}{-ikr}
\begin{pmatrix}
S_2 & S_3 \\
S_4 & S_1
\end{pmatrix}
\begin{pmatrix}
E_{vi} \\
E_{hi}
\end{pmatrix}
\] (B.1)

where \(y\) is the vertical direction in Cartesian coordinates, \(S_j\) are the complex amplitude functions, and \(r\) is the distance from the particle. The matrix containing the \(S_j\) is usually referred to as the scattering amplitude matrix, and the product of this matrix with the term preceding it in Eq. (B.1) is often called the Jones matrix (Hecht, 1987).

In terms of the Stokes parameters, the relationship between incident and scattered radiation is expressed as

\[
\begin{pmatrix}
I_s \\
Q_s \\
U_s \\
V_s
\end{pmatrix} = \frac{1}{k^2 r^2}
\begin{pmatrix}
S_{11} & S_{12} & S_{13} & S_{14} \\
S_{21} & S_{22} & S_{23} & S_{24} \\
S_{31} & S_{32} & S_{33} & S_{34} \\
S_{41} & S_{42} & S_{43} & S_{44}
\end{pmatrix}
\begin{pmatrix}
I_i \\
Q_i \\
U_i \\
V_i
\end{pmatrix}
\] (B.2)

where the \(S_{ij}\) matrix is the Stokes scattering matrix, also known as the Mueller matrix. The relationship given in Eq. (B.2) is simply a statement that says that the scattered Stokes parameters are a linear combination of the incident Stokes parameter. As explained by Chandrasekhar (1960), this is valid as long as there is no permanent phase relationship between the streams of energy in the system under consideration. This is equivalent to the statement that the distance between scatterers must be large compared to
wavelength, as specified in the opening statements of Chapter IV. The remainder of this appendix derives expressions for the $S_{ij}$ based on this assumption.

The Stokes parameters are defined in terms of the electromagnetic field as

$$I = \langle E_v E_v^* + E_h E_h^* \rangle$$ \hspace{1cm} (B.3a)

$$Q = \langle E_v E_v^* - E_h E_h^* \rangle$$ \hspace{1cm} (B.3b)

$$U = \langle E_v E_h^* + E_h E_v^* \rangle$$ \hspace{1cm} (B.3c)

$$V = i \langle E_v E_h^* - E_h E_v^* \rangle$$ \hspace{1cm} (B.3d)

where the angle brackets denote time-averaging as discussed in Chapter III. Using Eqs. (B.1) - (B.3), the subsequent sections develop relations between the Mueller and scattering amplitude matrix elements.

**B.2 Review of Complex Algebra**

Before proceeding to the development of the relationship between the Mueller and scattering amplitude matrix elements, a short review of complex algebra is presented. This is because the elements of the scattering amplitude matrix are complex numbers of the form

$$Z = X + i Y$$ \hspace{1cm} (B.4)

An alternate representation of $Z$ is given as

$$Z = R e^{i\theta} = R (\cos \theta + i \sin \theta)$$ \hspace{1cm} (B.5)

where

$$R = |Z| = (X^2 + Y^2)^{1/2}$$ \hspace{1cm} (B.6)

and

$$\theta = \tan^{-1} (Y / X)$$ \hspace{1cm} (B.7)

The real and imaginary parts of $Z$ are represented as
\[ \text{Re}(Z) = X = R \cos \theta \]  
(B.8)

\[ \text{Im}(Z) = Y = \dot{R} \sin \theta \]  
(B.9)

The complex conjugate of \( Z \) is written as

\[ Z^* = X - iY = R \ e^{-i\theta} = R (\cos \theta - i \sin \theta) \]  
(B.10)

Finally, some useful identities are

\[ (a + b)^* = a^* + b^* \]  
(B.11)

\[ (a \ b)^* = a^* \ b^* \]  
(B.12)

\[ \text{Re}(a \ b^*) = \text{Re}(b \ a^*) \]  
(B.13)

\[ \text{Im}(a \ b^*) = -\text{Im}(b \ a^*) \]  
(B.14)

\[ a \ b^* = b^* \ a \]  
(B.15)

\[ a + a^* = 2 \text{Re}(a) \]  
(B.16)

\[ a - a^* = 2 \text{Im}(a) \]  
(B.17)

### B.3 Mueller Matrix

By adding and subtracting pairs of Stokes parameters, Eq. (B.3) can be manipulated to obtain

\[ E_{\text{vi}} E_{\text{vi}}^* = \frac{1}{2} (I_i + Q_i) \]  
(B.18a)

\[ E_{\text{hi}} E_{\text{hi}}^* = \frac{1}{2} (I_i - Q_i) \]  
(B.18b)

\[ E_{\text{vi}} E_{\text{hi}}^* = \frac{1}{2} (U_i - V_i) \]  
(B.18c)

\[ E_{\text{hi}} E_{\text{vi}}^* = \frac{1}{2} (U_i + i \ V_i) \]  
(B.18d)

Also, expanding Eq. (B.1) gives

\[ E_{\text{vs}} = S_2 \ E_{\text{vi}} + S_3 \ E_{\text{hi}} \]  
(B.19a)

\[ E_{\text{hs}} = S_4 \ E_{\text{vi}} + S_1 \ E_{\text{hi}} \]  
(B.19b)

So, using Eq. (B.19a) and the identities from Eqs. (B.11) and (B.12)
\[ E_{vs} E_{vs}^* = [S_2 E_{vi} + S_3 E_{hi}] [S_2^* E_{vi}^* + S_3^* E_{hi}^*] \]
\[ = S_2 S_2^* + E_{vi} E_{vi}^* + S_2 S_3^* E_{vi} E_{hi}^* + S_3 S_2^* E_{hi} E_{vi}^* + S_3 S_3^* + E_{hi} E_{hi}^* \]
\[ = \frac{1}{2} \left[ |S_2|^2 (i_i + Q_i) + S_2 S_3^* (U_i - i V_i) + S_3 S_2^* (U_i + i V_i) + |S_3|^2 (i_i - Q_i) \right] \]

(B.20)

where Eq. (B.18) is used to introduce the Stokes parameters. Similarly,

\[ E_{hs} E_{hs}^* = [S_4 E_{vi} + S_1 E_{hi}] [S_4^* E_{vi}^* + S_1^* E_{hi}^*] \]
\[ = S_4 S_4^* + E_{vi} E_{vi}^* + S_1 S_4^* E_{hi} E_{vi}^* + S_4 S_1^* E_{vi} E_{hi}^* + S_1 S_1^* + E_{hi} E_{hi}^* \]
\[ = \frac{1}{2} \left[ |S_4|^2 (i_i + Q_i) + S_1 S_4^* (U_i + i V_i) + S_4 S_1^* (U_i - i V_i) + |S_1|^2 (i_i - Q_i) \right] \]

(B.21)

Now, using Eq. (B.3a)

\[ I_s = \frac{1}{2} \left[ |S_2|^2 (i_i + Q_i) + S_2 S_3^* (U_i - i V_i) + S_3 S_2^* (U_i + i V_i) + |S_3|^2 (i_i - Q_i) \right] \]
\[ + \frac{1}{2} \left[ |S_4|^2 (i_i + Q_i) + S_1 S_4^* (U_i + i V_i) + S_4 S_1^* (U_i - i V_i) + |S_1|^2 (i_i - Q_i) \right] \]

or

\[ I_s = \frac{1}{2} |S_2|^2 i_i + \frac{1}{2} |S_3|^2 i_i + \frac{1}{2} |S_4|^2 i_i + \frac{1}{2} |S_1|^2 i_i \]
\[ + \frac{1}{2} |S_2|^2 Q_i - \frac{1}{2} |S_3|^2 Q_i + \frac{1}{2} |S_4|^2 Q_i - \frac{1}{2} |S_1|^2 Q_i \]
\[ + \frac{1}{2} S_2 S_3^* U_i + \frac{1}{2} S_3 S_2^* U_i + \frac{1}{2} S_1 S_4^* U_i + \frac{1}{2} S_4 S_1^* U_i \]
\[ - \frac{1}{2} S_2 S_3^* V_i + \frac{1}{2} S_3 S_2^* V_i + \frac{1}{2} S_1 S_4^* V_i - \frac{1}{2} S_4 S_1^* V_i \]

But, from Eq. (B.2),

\[ I_s = S_{11} i_i + S_{12} Q_i + S_{13} U_i + S_{14} V_i \]

where

\[ S_{11} = \frac{1}{2} \left[ |S_1|^2 + |S_2|^2 + |S_3|^2 + |S_4|^2 \right] \]  

(B.22a)
\[ S_{12} = \frac{1}{2} \left[ |S_2|^2 - |S_1|^2 + |S_4|^2 - |S_3|^2 \right] \quad (B.22b) \]
\[ S_{13} = \frac{1}{2} \left[ S_2 S_3^* + S_3 S_2^* + S_1 S_4^* + S_4 S_1^* \right] \]
\[ = \frac{1}{2} \left[ \text{Re}(S_2 S_3^*) + \text{Im}(S_2 S_3^*) + \text{Re}(S_3 S_2^*) + \text{Im}(S_3 S_2^*) \right. \]
\[ + \text{Re}(S_1 S_4^*) + \text{Im}(S_1 S_4^*) + \text{Re}(S_4 S_1^*) + \text{Im}(S_4 S_1^*) \left. \right] \]
\[ = \text{Re} \left[ S_2 S_3^* + S_1 S_4^* \right] \quad (B.22c) \]
\[ S_{14} = \frac{1}{2} \left[ S_3 S_2^* - S_2 S_3^* + S_1 S_4^* - S_4 S_1^* \right] \]
\[ = \text{Im} \left[ S_2 S_3^* - S_1 S_4^* \right] \quad (B.22d) \]

The identities of Eqs. (B.11) - (B.14) have been used to simplify Eqs. (B.22c) and (B.22d). The remaining 12 elements of the Mueller matrix are obtained analogously, yielding

\[ S_{21} = \frac{1}{2} \left[ |S_2|^2 - |S_1|^2 - |S_4|^2 - |S_3|^2 \right] \quad (B.22e) \]
\[ S_{22} = \frac{1}{2} \left[ |S_2|^2 + |S_1|^2 - |S_4|^2 - |S_3|^2 \right] \quad (B.22f) \]
\[ S_{23} = \text{Re} \left[ S_2 S_3^* - S_1 S_4^* \right] \quad (B.22g) \]
\[ S_{24} = \text{Im} \left[ S_2 S_3^* + S_1 S_4^* \right] \quad (B.22h) \]
\[ S_{31} = \text{Re} \left[ S_2 S_4^* + S_1 S_3^* \right] \quad (B.22i) \]
\[ S_{32} = \text{Re} \left[ S_2 S_4^* - S_1 S_3^* \right] \quad (B.22j) \]
\[ S_{33} = \text{Re} \left[ S_1 S_2^* + S_3 S_4^* \right] \quad (B.22k) \]
\[ S_{34} = \text{Im} \left[ S_2 S_1^* + S_4 S_3^* \right] \quad (B.22l) \]
\[ S_{41} = \text{Im} \left[ S_2 S_4^* + S_3 S_1^* \right] \quad (B.22m) \]
\[ S_{42} = \text{Im} \left[ S_2 S_4^* - S_3 S_1^* \right] \quad (B.22n) \]
\[ S_{43} = \text{Im} \left[ S_1 S_2^* - S_3 S_4^* \right] \]
\[ S_{44} = \text{Re} \left[ S_1 S_2^* - S_3 S_4^* \right] \]

\[ \text{(B.22a,b)} \]

### B.4 Scattering Amplitude Matrix

In this section, expressions are developed that define the scattering amplitude matrix elements in terms of the Mueller matrix elements. Because the scattering amplitude elements for the Jones matrix are complex numbers of the form

\[ S_j = |S_j| \cos \theta_j + i |S_j| \sin \theta_j \]

the approach taken here is to find expressions for the scattering amplitude magnitudes (that is, the $|S_j|$) in terms of the Mueller matrix elements, and also to derive expressions for the scattering amplitude angles (the $\theta_j$). To find the magnitudes, pairs of Mueller matrix elements are added together. For example, using the results of the Section B.3,

\[ S_{11} + S_{12} = \frac{1}{2} \left[ |S_1|^2 + |S_2|^2 + |S_3|^2 + |S_4|^2 \right] \]

\[ + \frac{1}{2} \left[ |S_2|^2 - |S_1|^2 + |S_4|^2 - |S_3|^2 \right] \]

\[ = |S_2|^2 + |S_4|^2 \]

and

\[ S_{21} + S_{22} = \frac{1}{2} \left[ |S_2|^2 - |S_1|^2 - |S_4|^2 + |S_3|^2 \right] \]

\[ + \frac{1}{2} \left[ |S_2|^2 + |S_1|^2 - |S_4|^2 - |S_3|^2 \right] \]

\[ = |S_2|^2 - |S_4|^2 \]

Thus, \[ S_{11} + S_{12} + S_{21} + S_{22} = 2 |S_2|^2 \]. Solving for $|S_2|$ yields

\[ |S_2| = \sqrt{\frac{S_{11} + S_{21} + S_{22}}{2}} \]

\[ \text{(B.24a)} \]

Likewise, \[ S_{11} + S_{12} - S_{21} - S_{22} = 2 |S_4|^2 \], so
\[ |S_4| = \left[ \frac{S_{11} + S_{12} - S_{21} - S_{22}}{2} \right]^{1/2} \]  
(B.24b)

For \(|S_3|\), \(S_{11} - S_{12} + S_{21} - S_{22} = 2 |S_3|^2\), therefore

\[ |S_3| = \left[ \frac{S_{11} - S_{12} + S_{21} - S_{22}}{2} \right]^{1/2} \]  
(B.24c)

Finally, \(S_{11} - S_{12} - S_{21} + S_{22} = 2 |S_1|^2\), so that

\[ |S_1| = \left[ \frac{S_{11} - S_{12} + S_{21} - S_{22}}{2} \right]^{1/2} \]  
(B.24d)

The magnitude of the scattering amplitude matrix elements in terms of the Mueller matrix elements are given by Eq. (B.24). It should be pointed out that, in Eq. (B.24), the positive root is always taken.

To determine the angles needed for the scattering amplitude matrix elements, the procedure once again consists of adding various pairs of Mueller matrix elements:

\[ S_{13} + S_{23} = \text{Re} \left[ S_2 S_3^* + S_1 S_4^* \right] + \text{Re} \left[ S_2 S_3^* - S_1 S_4^* \right] \]

\[ = 2 \text{Re} \left[ S_2 S_3^* \right] \]

\[ = 2 \text{Re} \left[ \left( |S_2| e^{i\theta_2} \right) \left( |S_3| e^{-i\theta_3} \right) \right] \]

\[ = 2 |S_2| |S_3| \cos (\theta_2 - \theta_3) \]

or

\[ \cos (\theta_2 - \theta_3) = \frac{S_{13} + S_{23}}{2 |S_2| |S_3|} \]

Substituting from Eqs. (B.24a) and (B.24c) gives

\[ |S_2| |S_3| = \left[ \frac{S_{11} + S_{21} + S_{21} + S_{22}}{2} \right]^{1/2} \left[ \frac{S_{11} - S_{12} + S_{21} - S_{22}}{2} \right]^{1/2} \]
\[
\frac{\left[\left(S_{11} + S_{21}\right)^2 - \left(S_{11} + S_{21}\right)^2\right]}{2}^{1/2}
\]

so

\[
\cos (\theta_2 - \theta_3) = \frac{S_{13} + S_{23}}{\left[\left(S_{11} + S_{21}\right)^2 - \left(S_{11} + S_{21}\right)^2\right]^{1/2}}
\] (B.25a)

Also

\[
S_{14} + S_{24} = \text{Im} \left[ S_2 \ S_3^* - S_1 \ S_4^* \right] + \text{Im} \left[ S_2 \ S_3^* + S_1 \ S_4^* \right]
\]
\[
= 2 \ \text{Im} \left[ S_2 \ S_3^* \right]
\]
\[
= 2 \ \text{Im} \left[ (|S_2| e^{i\theta_2}) (|S_3| e^{-i\theta_3}) \right]
\]
\[
= 2 |S_2| |S_3| \sin (\theta_2 - \theta_3)
\]

so

\[
\sin (\theta_2 - \theta_3) = \frac{S_{14} + S_{24}}{2 |S_2| |S_3|}
\]
\[
= \frac{S_{14} + S_{24}}{\left[\left(S_{11} + S_{21}\right)^2 - \left(S_{11} + S_{21}\right)^2\right]^{1/2}}
\] (B.25b)

The remaining angles are found in the same manner:

\[
S_{31} + S_{32} = \text{Re} \left[ S_2 \ S_4^* - S_1 \ S_3^* \right] + \text{Re} \left[ S_2 \ S_4^* - S_1 \ S_3^* \right]
\]
\[
= 2 \ \text{Re} \left[ S_2 \ S_4^* \right]
\]
\[
= 2 \ \text{Re} \left[ (|S_2| e^{i\theta_2}) (|S_4| e^{-i\theta_4}) \right]
\]
\[
= 2 |S_2| |S_4| \cos (\theta_2 - \theta_4)
\]

so

\[
\cos (\theta_4 - \theta_2) = \frac{S_{31} + S_{32}}{\left[\left(S_{11} + S_{12}\right)^2 - \left(S_{21} + S_{22}\right)^2\right]^{1/2}}
\] (B.26a)

and

\[
S_{41} + S_{42} = \text{Im} \left[ S_2^* S_4 + S_3^* S_1 \right] + \text{Im} \left[ S_2^* S_4 - S_3^* S_1 \right]
\]
\[= 2 \text{Im} [S_2^* S_4]\]

\[= 2 \text{Im} \left[ (|S_2| e^{-i\theta_2}) (|S_4| e^{i\theta_4}) \right]\]

\[= 2 |S_2| |S_4| \sin (\theta_4 - \theta_2)\]

so

\[
\sin (\theta_4 - \theta_2) = \frac{S_{41} + S_{42}}{[(S_{11} + S_{12})^2 - (S_{21} + S_{22})^2]^{1/2}} \quad \text{(B.26b)}
\]

Finally,

\[S_{33} + S_{44} = \text{Re} \left[ S_1 S_2^* - S_3 S_4^* \right] + \text{Re} \left[ S_1 S_2^* - S_3 S_4^* \right]\]

\[= 2 \text{Re} [S_1 S_2^*]\]

\[= 2 \text{Re} \left[ (|S_1| e^{i\theta_1}) (|S_2| e^{-i\theta_2}) \right]\]

\[= 2 |S_1| |S_2| \cos (\theta_1 - \theta_2)\]

so

\[
\cos (\theta_1 - \theta_2) = \frac{S_{33} + S_{34}}{[(S_{11} + S_{22})^2 - (S_{21} + S_{12})^2]^{1/2}} \quad \text{(B.27a)}
\]

and

\[S_{43} - S_{34} = \text{Im} \left[ S_1 S_2^* - S_3 S_4^* \right] - \text{Im} \left[ S_2 S_1^* + S_4 S_3^* \right]\]

\[= \text{Im} \left[ S_1 S_2^* - S_3 S_4^* \right] + \text{Im} \left[ S_1 S_2^* + S_3 S_4^* \right]\]

\[= 2 \text{Im} [S_1 S_2^*]\]

\[= 2 \text{Im} \left[ (|S_1| e^{i\theta_2}) (|S_2| e^{-i\theta_2}) \right]\]

\[= 2 |S_1| |S_2| \sin (\theta_1 - \theta_2)\]

so

\[
\sin (\theta_1 - \theta_2) = \frac{S_{43} - S_{34}}{[(S_{11} + S_{22})^2 - (S_{21} + S_{12})^2]^{1/2}} \quad \text{(B.27b)}
\]

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The angles for the scattering amplitude matrix elements in terms of Mueller matrix elements are given completely by Eqs. (B.25) - (B.27). In these equations, the positive root is always taken.

The procedure for computing the scattering matrix elements $S_j$ from the Mueller matrix elements is as follows:

1. $\theta_2$ may be chosen arbitrarily. As noted by Gerrard and Burch (1975), this is the same as adding a time constant $\omega t$ to Eq. (B.1), which has no effect on any intensities.

2. Equation (B.25) determines $\theta_1 - \theta_2$ uniquely in the range $[0, 2\pi]$.

3. Equation (B.26) determines $\theta_4 - \theta_2$ uniquely in the range $[0, 2\pi]$.

4. Equation (B.27) determines $\theta_2 - \theta_3$ uniquely in the range $[0, 2\pi]$.

5. All the $\theta_j$ are known. Equation (B.24) is now used to compute $|S_j|$.


Finally, it should be pointed out that, as noted by Gerrard and Burch (1975), for every possible scattering amplitude matrix there corresponds a Mueller matrix. The converse, however, is not true. For example, if Eqs. (B.24) - (B.27) give imaginary values, or if the squares of the sine and cosine terms of each of Eqs. (B.25) - (B.27) do not sum to unity, then the given Mueller matrix represents a physical impossibility.

**B.5 Example**

As given by Hecht (1987), the Mueller matrix for a linear horizontal polarizer is given by

$$\overline{M} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The scattered Stokes vector for a unit-irradiance unpolarized wave passing through such a polarizer is given by
\[ I_s = \bar{m} \bar{I}_i = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \]

This equation states that the scattered wave has an intensity of \( I_s = 1/2 \), and that it is linearly polarized horizontally (\( Q_s > 0 \)). Now, to determine the scattering amplitude matrix for this polarizer, the procedure outlined at the end of Section B.4 is applied:

1. Let \( \theta_2 = 0 \).
2. Equation (B.25) gives \( \theta_1 - \theta_2 = 0 \).
3. Equation (B.26) gives \( \theta_4 - \theta_2 = 0 \).
4. Equation (B.27) gives \( \theta_2 - \theta_3 = 0 \).
5. Equation (B.24) gives \( |S_2| = 1 \), \( |S_1| = |S_3| = |S_4| = 0 \).
6. Equation (B.23) gives \( S_2 = 1 \), \( S_1 = S_3 = S_4 = 0 \).

In other words, the scattering amplitude matrix for a linear horizontal polarizer is

\[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

This agrees with the matrix given by Hecht (1987).
APPENDIX C.
VECTOR RADIATIVE TRANSFER EQUATION ELEMENTS

C.1 Introduction

The derivation of the vector radiative transfer equation proceeds by using the same methodology as is typically used for the scalar radiative transfer equation (for the scalar RTE, see Modest, 1993). However, for the former, the four Stokes parameters are used, while for the latter, only the intensity I is of interest. In general, the derivation begins by considering an infinitesimal element of solid angle $d\Omega$ and length $ds$, where an electromagnetic wave described by the Stokes parameter $\mathbf{T} = (I, Q, U, V)^T$ is incident normal to the lower surface of the element (Fig. C.1). The difference between the energy leaving the element in the direction normal to its upper face and the incident energy is described by counting up gains and losses along the path and can be written

$$dT = T(s+ds) - T(s)$$

$$= dT(\text{emission}) + dT(\text{inward scattering})$$

$$- dT(\text{absorption}) - dT(\text{outward scattering})$$ (C.1)

Expressions for the terms in Eq. (C.1) are developed in the subsequent sections.

C.2 Scattering

The relationship between incident and scattered Stokes parameters is given by Eq. (3.21) in terms of the Mueller matrix, where it is assumed that the scatterers are independent; that is, they are spaced far enough apart so as not to create any permanent electromagnetic phase relationship amongst themselves. When this is the case, the scattered Stokes parameters are simply a linear combination of the incident Stokes
Figure C.1 Radiation transfer for an element of solid angle $d\Omega$. 
parameters. Thus, the differential change in Stokes parameters due to scattering into direction \((\mu, \phi)\) of an incoming pencil of radiation of solid angle \(d\Omega\) arriving from direction \((\mu', \phi')\) is written by analogy with the scalar RTE as

\[
d\mathbf{T} \text{ (inward scattering)} = \int_0^{2\pi} \int_{-1}^1 \mathbf{Z}(\mu, \phi; \mu', \phi') \mathbf{T}(\mu', \phi') \, d\mu' \, d\phi' \, ds
\]

where the geometry for the polarization reference frame is given by Fig. 3.4. In Eq. (C.2), the rotated scattering phase matrix has been used as explained in Section 3.5 and in particular Eq. (3.28). The explicit form of Eq. (3.28) is given by Takano and Liou (1993), and is reproduced here for convenience (their element \(Z_{21}\) contains a typographical error that is corrected here):

\[
\mathbf{Z}(\mu, \phi; \mu', \phi') = \mathbf{L}(\pi - i_2) \mathbf{F}(\mu, \phi; \mu', \phi') \mathbf{L}(-i_1) = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{pmatrix}
\]

where the submatrices \(\overline{A}_{jk}\) are written as

\[
\overline{A}_{11} = \begin{pmatrix} P_{11} & P_{12}C' + P_{13}S' \\ P_{21}C - P_{31}S & P_{22}CC' - P_{32}SC' + P_{23}CS' - P_{33}SS' \end{pmatrix}
\]

\[
\overline{A}_{12} = \begin{pmatrix} -P_{12}S' + P_{13}C' \\ -P_{22}CS' + P_{32}SS' + P_{23}CC' - P_{33}SC' \end{pmatrix}
\]

\[
\overline{A}_{21} = \begin{pmatrix} P_{21}S + P_{31}C \\ P_{41} \end{pmatrix}
\]

\[
\overline{A}_{22} = \begin{pmatrix} P_{22}CS' + P_{32}CC' + P_{23}SS' + P_{33}CC' \\ -P_{42}S' + P_{43}C' \end{pmatrix}
\]

and where the following abbreviations have been used:
\( S' = \sin(2i_1), \quad S = \sin(2i_2), \quad C' = \cos(2i_1), \quad C = \cos(2i_2) \) (C.5)

### C.3 Extinction

Extinction reduces the radiation traversing a medium, and is due to absorption and outward scattering along a path, that is,

\[
d\bar{T} \text{(extinction)} = d\bar{T} \text{(absorption)} + d\bar{T} \text{(outward scattering)} \tag{C.6}
\]

To describe the extinction in terms of the Stokes parameters, it is necessary to introduce the differential field equations (Ishimaru and Cheung, 1980) that describe the change of the electric field along a path \( ds \), and are given by

\[
\frac{dE_v}{ds} = M_{vv} E_v + M_{vh} E_h \tag{C.7a}
\]

\[
\frac{dE_h}{ds} = M_{hv} E_v + M_{hh} E_h \tag{C.7b}
\]

where the \( M_{\alpha\beta} \) are defined in Eq. (4.24). The procedure to find the change in the Stokes parameters along the path \( ds \) begins by differentiating Eq. (3.5) with respect to \( ds \) to obtain

\[
\frac{dI}{ds} = \frac{1}{2\eta} \frac{d}{ds} \left[ E_v^* E_v^* + E_h^* E_h^* \right]
\]

\[
= \frac{1}{2\eta} \left[ E_v^* \frac{d}{ds} E_v + E_v \frac{d}{ds} E_v^* + E_h^* \frac{d}{ds} E_h + E_h \frac{d}{ds} E_h^* \right] \tag{C.8}
\]

Substitution of Eq. (C.7) into Eq. (C.8) gives

\[
\frac{dI}{ds} = M_{vv} E_v E_v^* + M_{vh} E_h E_v^* + M_{vv} E_v E_v^* + M_{vh} E_h E_h^* E_v
\]

\[
+ M_{hv} E_v E_h^* + M_{hh} E_h E_h^* + M_{hv} E_v E_h^* E_h + M_{hh} E_h E_h^* E_h \tag{C.9}
\]

Insertion of Eqs.(3.5) through (3.8) into Eq. (C.9) gives

\[
\frac{dI}{ds} = \frac{1}{2} M_{vv} (I + Q) + \frac{1}{2} M_{vh} (U + i V) + \frac{1}{2} M_{vv} (I + Q) + \frac{1}{2} M_{vh} (U + i V)
\]

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\[ + \frac{1}{2} M_{hv} (U - i V) + \frac{1}{2} M_{hh} (I - Q) + \frac{1}{2} M_{hv}^* (U - i V) + \frac{1}{2} M_{hh}^* (I - Q) \]  

or

\[ \frac{\text{d}I}{\text{d}s} = \frac{1}{2} \left( M_{vv} + M_{hh} + \frac{1}{2} M_{hv} \right) + \frac{1}{2} Q \left( M_{vv} + M_{hh} - \frac{1}{2} M_{hv} \right) \]

\[ + \frac{1}{2} U \left( M_{vh} + M_{hv} \right) + \frac{1}{2} M_{hv}^* \]

\[ + \frac{i}{2} V \left( M_{vh} + M_{hv}^* - \frac{1}{2} M_{hv} \right) \]

which can be rewritten as

\[ \frac{\text{d}I}{\text{d}s} = \text{Re} (M_{vv} + M_{hh}) I + \text{Re} (M_{vv} - M_{hh}) Q \]

\[ + \text{Re} (M_{vh} + M_{hv}) U + \text{Im} (M_{vh} - M_{hv}) V \]  

In vector notation, Eq. (C.12) is written as

\[ \frac{\text{d}I}{\text{d}s} = -\kappa_{e,1} \overline{I} \]  

where \( \kappa_{e,1} \) is the top row of the extinction matrix as given in Eq. (4.23). The minus sign in Eq. (C.13) has been introduced for convenience to make it clear that the extinction term describes a loss, and also so that it has the same form as the extinction term in the scalar RTE. The procedure used in Eqs. (C.8) through (C.12) is applied to the remaining Stokes parameters \( Q, U, \) and \( V \) to obtain rows two through four of the extinction matrix. As a result, Eq. (3.5) may be written as

\[ \text{d} \overline{T} \text{(extinction)} = -\overline{\kappa}_e \overline{T} \text{ ds} \]  

\[ \text{C.4 Emission} \]

Kirchoff’s law states that, for a body in local thermodynamic equilibrium, spectral directional absorptance is equal to spectral directional emittance. In the scalar RTE, absorption of energy is described by the absorption coefficient \( k_a \), and therefore, due to
Kirchoff's law, the proportionality constant for emission of energy is also described by $k_a$. To maintain consistency between the notation used in the RTE and VRTE, the emission vector used in the latter is given the subscript "a". Using conservation of energy arguments, Tsang (1984) derived an expression for the emission vector in terms of the extinction and scattering phase matrices as

$$
\bar{k}_{a*} = \begin{pmatrix}
  k_{e11*} - \int_{4\pi} [P_{11*} + P_{21*}] \, d\Omega' \\
  k_{e22*} - \int_{4\pi} [P_{12*} + P_{22*}] \, d\Omega' \\
  2k_{e13*} + 2k_{e23*} - 2 \int_{4\pi} [P_{13*} + P_{23*}] \, d\Omega' \\
  -2k_{e14*} - 2k_{e24*} + 2 \int_{4\pi} [P_{14*} + P_{24*}] \, d\Omega'
\end{pmatrix}
$$

(C.15)

where the subscript * refers to the alternate Stokes basis ($I_v$, $I_h$, $U$, $V$). For unpolarized radiation, the absorption coefficient used in the RTE obeys the relation

$$
k_a = k_e - k_s
$$

(C.16)

Thus, for unpolarized radiation, Eq. (C.15) should reduce to Eq. (C.16), assuming that the former has been transformed to the standard Stokes basis ($I$, $Q$, $U$, $V$). However, this is not the case. Evans and Stephens (1993) provide an alternate expression for the emission vector, and this has been reproduced in the text as Eq. (4.25). Evans and Stephens (1993) also check their emission vector values by computing power dissipation and ensuring that it agrees with the expression given by Eq. (C.15). In any case,

$$
dT(\text{emission}) = \bar{k}_a \, I_B \, ds
$$

(C.17)

where $I_B$ is the blackbody intensity, given by Eq. (3.40).

C.5 Summary

Expressions that describe the differential change in the Stokes vector due to inward scattering, extinction (outward scattering and absorption), and emission, are given by Eqs. (C.2), (C.14), and (C.17), respectively. Substitution of these equations into Eq. (C.1) yields Eq. (3.26) as desired.
APPENDIX D.
EXAMPLES

This appendix contains examples of various topics referred to in the text: Section D.1 transforms vectors and matrices using different Stokes bases, Section D.2 gives examples of various forms of the scattering phase matrix, Section D.3 gives examples of extinction matrices, and Section D.4 gives a solution to the VRTE for a non-scattering medium over a specular surface with thermal sources. Some of the symbols used are unique to Appendix D and are not included in the Nomenclature.

D.1 Stokes Basis Transformations

Section 3.6 gives expressions for transforming vectors and matrices from the Stokes basis to the modified Stokes basis and vice versa. As an example, consider the emission vectors for the Stokes and modified Stokes bases, expressed as

\[
\begin{align*}
\overline{k}_a &= \begin{pmatrix} \kappa_{a1} \\ \kappa_{a2} \\ \kappa_{a3} \\ \kappa_{a4} \end{pmatrix}, & \overline{k}_{a*} &= \begin{pmatrix} \kappa_{a*1} \\ \kappa_{a*2} \\ \kappa_{a*3} \\ \kappa_{a*4} \end{pmatrix} \\
\end{align*} \tag{D.1}
\]

Application of Eq. (3.36) to Eq. (D.1) allows \( \overline{k}_a \) and \( \overline{k}_{a*} \) to be expressed in terms of each other as

\[
\begin{align*}
\overline{k}_a &= \begin{pmatrix} \kappa_{a*1} + \kappa_{a*2} \\ \kappa_{a*1} - \kappa_{a*2} \\ \kappa_{a*3} \\ \kappa_{a*4} \end{pmatrix}, & \overline{k}_{a*} &= \begin{pmatrix} \frac{1}{2} (\kappa_{a1} + \kappa_{a2}) \\ \frac{1}{2} (\kappa_{a1} - \kappa_{a2}) \\ \kappa_{a3} \\ \kappa_{a4} \end{pmatrix} \\
\end{align*} \tag{D.2}
\]

As a second example, application of Eq. (3.31) can be applied to the extinction matrix for the modified Stokes basis to yield an expression for the extinction matrix for the standard Stokes basis written in partitioned matrix form as
\[ \mathbf{\bar{k}}_e = \begin{pmatrix} \mathbf{\bar{A}}_{11} & \mathbf{\bar{A}}_{12} \\ \mathbf{\bar{A}}_{21} & \mathbf{\bar{A}}_{22} \end{pmatrix} \]  

where the submatrices \( \mathbf{\bar{A}}_{jk} \) are

\[ \mathbf{\bar{A}}_{11} = \frac{1}{2} \begin{pmatrix} (k_{e_{11}} + k_{e_{21}} + k_{e_{12}} + k_{e_{22}}) & (k_{e_{11}} + k_{e_{21}} - k_{e_{12}} - k_{e_{22}}) \\ (k_{e_{11}} - k_{e_{21}} + k_{e_{12}} - k_{e_{22}}) & (k_{e_{11}} - k_{e_{21}} - k_{e_{12}} + k_{e_{22}}) \end{pmatrix} \]  

\[ \mathbf{\bar{A}}_{12} = \begin{pmatrix} k_{e_{13}} + k_{e_{23}} & k_{e_{14}} + k_{e_{24}} \\ k_{e_{13}} - k_{e_{23}} & k_{e_{14}} - k_{e_{24}} \end{pmatrix} \]  

\[ \mathbf{\bar{A}}_{21} = \frac{1}{2} \begin{pmatrix} k_{e_{31}} + k_{e_{32}} & k_{e_{31}} - k_{e_{32}} \\ k_{e_{41}} + k_{e_{42}} & k_{e_{41}} - k_{e_{42}} \end{pmatrix} \]  

\[ \mathbf{\bar{A}}_{22} = \begin{pmatrix} k_{e_{33}} & k_{e_{34}} \\ k_{e_{43}} & k_{e_{44}} \end{pmatrix} \]

Likewise, the extinction matrix in the modified Stokes basis can be written in terms of the extinction matrix in the standard Stokes basis as

\[ \mathbf{\bar{k}}_{e*} = \begin{pmatrix} \mathbf{\bar{A}}_{11} & \mathbf{\bar{A}}_{12} \\ \mathbf{\bar{A}}_{21} & \mathbf{\bar{A}}_{22} \end{pmatrix} \]  

where the submatrices \( \mathbf{\bar{A}}_{jk} \) are

\[ \mathbf{\bar{A}}_{11} = \frac{1}{2} \begin{pmatrix} (k_{e_{11}} + k_{e_{21}} + k_{e_{12}} + k_{e_{22}}) & (k_{e_{11}} + k_{e_{21}} - k_{e_{12}} - k_{e_{22}}) \\ (k_{e_{11}} - k_{e_{21}} + k_{e_{12}} - k_{e_{22}}) & (k_{e_{11}} - k_{e_{21}} - k_{e_{12}} + k_{e_{22}}) \end{pmatrix} \]  

\[ \mathbf{\bar{A}}_{12} = \frac{1}{2} \begin{pmatrix} k_{e_{13}} + k_{e_{23}} & k_{e_{14}} + k_{e_{24}} \\ k_{e_{13}} - k_{e_{23}} & k_{e_{14}} - k_{e_{24}} \end{pmatrix} \]  

\[ \mathbf{\bar{A}}_{21} = \begin{pmatrix} k_{e_{31}} + k_{e_{32}} & k_{e_{31}} - k_{e_{32}} \\ k_{e_{41}} + k_{e_{42}} & k_{e_{41}} - k_{e_{42}} \end{pmatrix} \]
\[
\overline{A}_{22} = \begin{pmatrix} k_{e33} & k_{e34} \\ k_{e43} & k_{e44} \end{pmatrix}
\]  

(D.6d)

Expressions for the phase matrices in standard or modified Stokes basis are obtained by replacing the symbol "k" in the previous equations with "P" or "Z".

D.2 Special Forms of the Scattering Phase Matrix

Several special forms of the scattering phase matrix \( \overline{P} \) are presented in this section. In general, these matrices must be rotated as described in Section 3.5.2 to be used in the VRTE. Other special forms of the phase matrix are given by van de Hulst (1981).

D.2.1 Six Element

The six element phase matrix (van de Hulst, 1981; Hansen and Travis, 1974) can be written as

\[
\overline{P} = \begin{pmatrix} P_1 & P_2 & 0 & 0 \\ P_2 & P_3 & 0 & 0 \\ 0 & 0 & P_3 & -P_4 \\ 0 & 0 & P_4 & P_6 \end{pmatrix}
\]  

(D.7)

This form of the phase matrix is valid for: i) randomly oriented symmetric particles; ii) randomly oriented asymmetric particles with half the number of particles being mirror images of the others; and iii) Rayleigh and Mie scattering (see subsequent sections).

D.2.2 Spherical Particles

Scattering by an ensemble of spherical particles is commonly called Mie scattering. For this situation, the phase matrix is given by setting \( P_5 = P_1 \) and \( P_6 = P_3 \) in Eq. (D.7) so that the resulting matrix has four unique elements, that is

\[
\overline{P} = \begin{pmatrix} P_1 & P_2 & 0 & 0 \\ P_2 & P_1 & 0 & 0 \\ 0 & 0 & P_3 & -P_4 \\ 0 & 0 & P_4 & P_3 \end{pmatrix}
\]  

(D.8)
An interpretation of these quantities is offered by White (1979):

\( P_1 \equiv \) the phase function that describes the redistribution of radiant energy from an incoming direction to an outgoing direction. \( P_1 \) is always non-zero.

\( P_2 / P_1 \equiv \) the degree of polarization of scattered radiant energy for unpolarized incident radiant energy. For vertical polarization and Rayleigh scattering, \( P_2 < 0 \); for horizontal polarization, \( P_2 > 0 \).

\( P_3 / P_1 \equiv \) the "skew" polarization that represents linearly polarized radiant energy but at an angle to the scattering plane (that is, this quantity does not represent horizontal or vertical polarization).

\( P_4 / P_1 \equiv \) the degree of circular polarization: this is the fraction of skew-polarized radiant energy that is converted to circular polarization. If the electric vector rotates clockwise with respect to the emitter, then \( P_4 > 0 \); otherwise, \( P_4 \leq 0 \).

**D.2.3 Rayleigh Scattering**

Rayleigh scattering typically refers to the scattering of radiant energy by spherical particles that are much smaller than the wavelength being considered. For this situation, the phase matrix is written as

\[
\frac{\overline{P}}{P} = \frac{3}{4} \begin{pmatrix}
1 + \cos^2 \Theta & 1 - \cos^2 \Theta & 0 & 0 \\
1 - \cos^2 \Theta & 1 + \cos^2 \Theta & 0 & 0 \\
0 & 0 & 2 \cos \Theta & 0 \\
0 & 0 & 0 & 2 \cos \Theta
\end{pmatrix}
\]  

(D.9)

where \( \Theta \) is the scattering angle. However, experiments on scattering indicate that in practice radiant energy is not usually scattered in accordance with the phase matrix given by Eq. (D.9), presumably due to the anisotropy of real particles, for example, by gaseous molecules. However, this problem has been treated theoretically and is reviewed by Chandrasekhar (1960) and Hansen and Travis (1974), who state the phase matrix for Rayleigh scattering by anisotropic particles has the form
\[
\overline{P} = \frac{3\Delta}{4} \begin{pmatrix}
1 + \cos^2\Theta & 1 - \cos^2\Theta & 0 & 0 \\
1 - \cos^2\Theta & 1 + \cos^2\Theta & 0 & 0 \\
0 & 0 & 2\cos\Theta & 0 \\
0 & 0 & 0 & 2\Delta' \cos\Theta \\
\end{pmatrix}
\]

\[
+ (1 - \Delta) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\] (D.10)

where

\[
\Delta = \frac{1 - \delta}{1 + \delta/2}, \quad \Delta' = \frac{1 - 2\delta}{1 - \delta}
\] (D.11)

and \(\delta\) is the depolarization factor. A typical value of \(\delta\) for air molecules is about 0.03, but this value is wavelength dependent. The topic of anisotropic Rayleigh scattering is an active area of research, with much effort devoted to the determination of the depolarization factor, for example, see Bucholtz (1995).

D.2.4 Oriented Scatterers with Rotational Symmetry

For vertical incidence on scatterers with rotational symmetry (for example, columns) whose long axis is parallel to the horizontal plane and randomly oriented within this plane, Tang and Aidin (1995) show that the Mueller matrix is

\[
\overline{M} = \frac{1}{2} \begin{pmatrix}
P & 0 & 0 & 0 \\
0 & \frac{1}{2}P + R & 0 & 0 \\
0 & 0 & \frac{1}{2}P + R & 0 \\
0 & 0 & 0 & 2R \\
\end{pmatrix}
\] (D.12)

where

\[
P = |S_1|^2 + |S_2|^2
\] (D.13a)

\[
R = \text{Re}\{S_2S_1^*\}
\] (D.13b)
and $S_j$ are the scattering amplitude functions defined in Section 3.3.

**D.2.5 Expansion Coefficients**

For the six element phase matrix of Section 2.1, the phase matrix elements can be expanded in generalized spherical functions as (Kuik et al., 1992; Mishchenko, 1992a)

\[
P_1(\Theta) = \sum_{l=0}^{\infty} \alpha_1^l P_{0,0}^l(\cos \Theta)
\]

\[
P_5(\Theta) + P_3(\Theta) = \sum_{l=2}^{\infty} (\alpha_2^l + \alpha_3^l) P_{2,2}^l(\cos \Theta)
\]

\[
P_5(\Theta) - P_3(\Theta) = \sum_{l=2}^{\infty} (\alpha_2^l - \alpha_3^l) P_{2,2}^l(\cos \Theta)
\]

\[
P_6(\Theta) = \sum_{l=0}^{\infty} \alpha_4^l P_{0,0}^l(\cos \Theta)
\]

\[
P_2(\Theta) = \sum_{l=2}^{\infty} \beta_4^l P_{0,2}^l(\cos \Theta)
\]

\[
-P_4(\Theta) = \sum_{l=2}^{\infty} \beta_4^l P_{0,2}^l(\cos \Theta)
\]

where $P_{m,n}^l$ are the generalized spherical functions, and $l$, $m$, and $n$ are integers. The expansion coefficients are computed using

\[
\alpha_1^l = \left(1 + \frac{1}{2}\right) \int_{-1}^{1} P_{0,0}^l(\cos \Theta) d(\cos \Theta)
\]

\[
\alpha_2^l + \alpha_3^l = \left(1 + \frac{1}{2}\right) \int_{-1}^{1} P_{2,2}^l(\cos \Theta) (P_5(\Theta) + P_3(\Theta)) d(\cos \Theta)
\]
\[ \alpha_2^1 - \alpha_3^1 = \left(1 + \frac{1}{2}\right) \int_{-1}^{1} P_{2,-2}^1 \{P_5(\Theta) - P_3(\Theta)\} \, d(\cos \Theta) \] (D.22)

\[ \alpha_4^1 = \left(1 + \frac{1}{2}\right) \int_{-1}^{1} P_{0,0}^1 P_6(\Theta) \, d(\cos \Theta) \] (D.23)

\[ \beta_1^1 = \left(1 + \frac{1}{2}\right) \int_{-1}^{1} P_{0,2}^1 P_2(\Theta) \, d(\cos \Theta) \] (D.24)

\[ -\beta_2^1 = \left(1 + \frac{1}{2}\right) \int_{-1}^{1} P_{0,2}^1 P_4(\Theta) \, d(\cos \Theta) \] (D.25)

Typically, a scattering phase matrix for an ensemble of scatterers can be computed for 10-50 discrete scattering angles, and then the expansion coefficients can be computed using Eqs. (D.20) – (D.25). These expansion coefficients may then be used as inputs to a radiative transfer model, rather than the phase matrix elements. In principle, it should be possible to expand the general sixteen element phase matrix in terms of the generalized spherical functions. Note that, because the functions \(P_{0,0}^1\) are the Legendre polynomials, the expression for the first phase matrix element \(P_1\), Eq. (D.14), reduces to the commonly used Legendre polynomial expansion of the phase function

\[ \Phi(\Theta) = \sum_{l=0}^{\infty} \alpha_l \, P_l(\cos \Theta) \] (D.26)

with

\[ \alpha_l = \left(1 + \frac{1}{2}\right) \int_{-1}^{1} P_l \Phi(\Theta) \, d(\cos \Theta) \] (D.27)

and \(P_l\) are the Legendre polynomials of degree \(l\).
D.2.6 Parameterizations

In its most general form, the phase matrix contains sixteen elements, each of which are dependent on particle size, shape, orientation, composition, and on wavelength and direction of the incoming and outgoing beams. Clearly, for even modestly complex problems, storage and/or computation of the phase matrix can overwhelm even the most powerful computers. Thus, it is desirable to parameterize the phase matrix elements. For the scalar radiative transfer equation, only one element of the phase matrix is used, and many parameterization schemes are available, for example, the Heyney-Greenstein phase function (van de Hulst, 1980). If particles approach the Rayleigh scattering regime, an acceptable parameterization would be to use the phase matrices given in Section D.2.3. For larger particles, however, this parameterization would not be valid. A review of the literature found only two phase matrix parameterization schemes, both applicable to spherical particles, one by Hovenier (1971), and the other by White (1979). Though these parameterizations appear to be useful, it is not clear that they satisfy the relationships presented by Fry and Kattawar (1981), or the symmetry relationships given by Hovenier (1969). Evans and Stephens (1995a) performed least-squares fits to scattering phase matrix elements for aspherical ice crystals, but, although this is a significant contribution, the resulting coefficients only describe the particle size-shape distribution used in that study. Clearly, there is a need for further investigation to be performed in this area.

D.3 Special Forms of the Extinction Matrix

Some special forms of the extinction matrix \( \mathbf{k}_e \) are considered in this section. Much of this discussion is based on Mishchenko (1994b).

D.3.1 Diagonal

The diagonal extinction matrix is written
\[
\bar{k}_e = \begin{pmatrix}
  k_e & 0 & 0 & 0 \\
  0 & k_e & 0 & 0 \\
  0 & 0 & k_e & 0 \\
  0 & 0 & 0 & k_e \\
\end{pmatrix}
\]  

(Eq. D.28)

where \( k_e \) is the extinction coefficient that is commonly used for scalar radiative transfer. This form of the extinction matrix is valid for: i) isotropic media composed of randomly oriented particles having a plane of symmetry (for example, spheres and spheroids); and ii) randomly oriented asymmetric particles with half the number of particles being mirror images of the others.

**D.3.2 Three Element**

The three element extinction matrix has the form

\[
\bar{k}_e(\mu) = \begin{pmatrix}
  k_e(\mu) & k_p(\mu) & 0 & 0 \\
  k_p(\mu) & k_e(\mu) & 0 & 0 \\
  0 & 0 & k_e(\mu) & k_c(\mu) \\
  0 & 0 & -k_c(\mu) & k_e(\mu) \\
\end{pmatrix}
\]  

(Eq. D.29)

where \( k_e, k_p, \) and \( k_c \) are the coefficients of extinction, linear polarization, and circular polarization, respectively, and \( \mu \) is the cosine of the zenith angle of the outgoing beam. This form of the extinction matrix is valid for i) aspherical particles randomly oriented in the horizontal plane; and ii) aspherical axially oriented particles.

**D.3.3 Parameterizations**

A review of the literature for extinction matrix parameterizations produced only the work of Evans and Stephens (1995a), who performed least-squares fits to \( 2 \times 2 \) extinction matrix elements for aspherical ice crystals. The coefficients computed in that study describe a particular particle size-shape distribution at millimeter wavelengths. Kummerow and Weinman (1988a) provide parameterizations of the scalar extinction
coefficient for aspherical hydrometeors at commonly used microwave frequencies, but do not consider the extinction matrix. Evans and Fournier (1994) developed an analytic approximation for the extinction efficiency factor applicable to randomly oriented spheroids, but this is only useful for computation of the scalar extinction coefficient.

D.4 Solution to the VRTE for a Non-Scattering Medium over a Specular Surface with Thermal Sources

The following solution to the VRTE is from Ulaby et al. (1981, Vol. I, p. 230), and is valid for a plane-parallel non-scattering medium over a specular surface with thermal sources. For these conditions, the U and V Stokes parameters vanish, and the equations for I and Q are completely uncoupled. In addition, the extinction matrix is diagonal so that \( k_{e11} = k_{e22} = k_a \). The solution for the VRTE is given in terms of brightness temperature, and thus is valid only when the Rayleigh-Jeans law applies (for example, in the microwave). However, the solution is completely general if temperatures are replaced by the appropriate radiant intensities. For an observation angle \( \theta \), surface height \( H \), and polarization state \( p \), the solution is given by

\[
T_B(\theta; H; p) = \frac{[1 - \Gamma(\theta; p)] T_g + \Gamma(\theta; p) T_{DN}(\theta)}{L_a(\theta; H)} + T_{UP}(\theta; H)
\]  

(D.30)

where \( \Gamma(\theta; p) \) are Fresnel reflectivities and are defined by

\[
\Gamma(\theta; h) = |R_h|^2
\]  

(D.31a)

\[
\Gamma(\theta; v) = |R_v|^2
\]  

(D.31b)

and the Fresnel reflection coefficients are defined in Eq. (4.31). The atmospheric loss factor is given by

\[
L_a(\theta; H) = e^{-\tau(0,H) \sec \theta}
\]  

(D.32)
where $\tau(a,b) = \int_a^b k_a$ is optical thickness. The upward and downward atmospheric apparent temperatures used in Eq. (D.30) are given respectively by

$$T_{DN}(\theta) = \sec\theta \int_0^\infty k_a(z) T(z) e^{-\tau(0,z)\sec\theta} \, dz$$  \hspace{1cm} (D.33a)$$

$$T_{UP}(\theta; H) = \sec\theta \int_0^H k_a(z) T(z) e^{-\tau(z,H)\sec\theta} \, dz$$  \hspace{1cm} (D.33b)$$

where $z$ is altitude. For the special case of a isothermal homogeneous atmosphere, $k_a$ and $T$ are constants, and Eq. (D.33) reduces to

$$T_{DN}(\theta) = T \left[ 1 - e^{-\tau(0,\infty)\sec\theta} \right]$$  \hspace{1cm} (D.34a)$$

$$T_{UP}(\theta; H) = T \left[ 1 - e^{-\tau(z,H)\sec\theta} \right]$$  \hspace{1cm} (D.34b)$$