FACTORS OF SAFETY FOR RICHARDSON EXTRAPOLATION
FOR INDUSTRIAL APPLICATIONS

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List of Nomenclature

$C_k$ = correction factor
$\varepsilon$ = solution change
$Fr$ = Froude number
$FS$ = factors of safety
$p$ = order of accuracy
$R$ = solution ratio
$r$ = refinement ratio
$S$ = solution
$U$ = uncertainty; velocity
$y^+$ = wall coordinate ($= U_\tau y / \nu$)
$\delta_{RE_{th}}$ = error
$\nu$ = kinematic viscosity

Subscripts

1 = first grid point away from the wall
1,2,3 = fine, medium, and coarse grids, respectively
$C$ = corrected
$G$ = grid
$k$ = the $k^{th}$ parameter
$RE$ = Richardson Extrapolation
$th$ = theoretical
$\tau$ = frictional
Abstract

Improved factors of safety for quantitative estimates for grid and time convergence uncertainties for CFD solutions are proposed for situations when Richardson extrapolation estimated order of accuracy $p_k$ is larger than the theoretical order of accuracy $p_{k_\text{th}}$ and correction factor $1 < C_k < 2$. The improved uncertainty estimates are shown to provide more reasonable intervals of uncertainty for $p_k > p_{k_\text{th}}$ ($1 < C_k < 2$).
I. INTRODUCTION

Current procedures for quantitative estimates for grid and time convergence uncertainties for CFD solutions are based on first-order Richardson extrapolation (RE) error estimates with factors of safety ($FS$) used for expanded uncertainty estimates. Roache [1,2] proposed the grid-convergence index (GCI) with $FS=1.25$ for systematic grid-triplet studies using RE estimate for order of accuracy $p_k$ ($k=G$ for grid, $k=T$ for time, and $k=P$ for similar parameters) and $FS=3$ for 2-grid sensitivity studies using theoretical estimate for order of accuracy $p_{kh}$. The GCI is widely used and recommended by ASME [3] and AIAA [4]. The authors and colleagues proposed a correction factor ($C_k$) method [5,6] with linearly increasing $FS$ vs. distance from the asymptotic range (AR) $C_k = (r^{p_k} - 1)/(r^{p_{kh}} - 1)$, which was based on analytical benchmarks that approach the AR with $p_k < p_{kh}$. $FS$ was reflected for $p_k > p_{kh}$. $FS$ using $C_k$ is smaller than $FS=1.25$ for solutions very close to the AR, whereas $FS$ using $C_k$ is much larger than $FS=1.25$ for solutions far from the AR, which is the typical situation for industrial applications. $C_k$ has the “common-sense” advantage compared to GCI in providing a quantitative metric to determine proximity of the solutions to the AR and approximately accounts for the effects of higher-order RE terms. The $C_k$ method has been used in ship hydrodynamics CFD workshops.

A deficiency of both GCI and $C_k$ methods is that for $p_k > p_{kh}$ the uncertainty estimates are unreasonably small in comparison to uncertainty estimates for $p_k < p_{kh}$ with similar grid refinement ratio $r_k$, solution changes between fine and medium grid/time $\varepsilon_{k_2}$ and distance $|p_k - p_{kh}|$ from the AR, which results from too small RE error estimate:

$$\delta_{RE_{n}}^* = \frac{\varepsilon_{k_2}}{r_k^{p_k} - 1}$$ (1)
In the GCI method $FS=1.25$ is much too small and in the $C_k$ method linearly increasing $FS$ is also too small. Herein, an improvement to the $C_k$ method is proposed with polynomial increasing $FS$ for $p_k > p_{k_0}$ based on reflecting the grid/time uncertainty from $p_k < p_{k_0}$ for $p_k > p_{k_0}$. The improved uncertainty estimates are shown to provide more reasonable intervals of uncertainty for $p_k > p_{k_0}$.

II. IMPROVED FS FOR RE ESTIMATED LARGER THAN THEORETICAL ORDER OF ACCURACY

Grid and time convergence studies are conducted with multiple solutions (at least 3) using systematically refined grid sizes or time steps. For monotonic convergence, procedures for estimating grid and time errors are based on RE, which assumes that the error terms are in the form of power series expansion. Results from the numerical solution of the one-dimensional wave and two-dimensional Laplace equation analytical benchmarks show that RE error estimate using Eqn. (1) has the right form/trends, but is only qualitatively not quantitatively accurate due to poorly estimated $p_k < p_{k_0}$.

$$p_k = \frac{\ln\left(\frac{\epsilon_{k_0}}{\epsilon_{k_1}}\right)}{\ln(r_k)}$$

except when solutions are very close to the AR. The error estimate can be improved using correction factors, i.e., $\delta_k^* = C_k \delta_{RE_i}$, which is used to estimate the uncertainty for uncorrected solutions by bounding the error $\delta_k^*$ by the sum of the absolute value of the corrected RE error estimate and the absolute value of the amount of the correction with provision for 10% FS in the limit of $C_k = 1$. $FS = U_k/\left|\delta_{RE_i}\right|$ is reflected from $p_k < p_{k_0}$ for $p_k > p_{k_0}$ [6]. Thus for uncorrected solutions,
For corrected solutions, $U_{k_c}$ is based on the absolute value of the amount of the correction:

$$
U_{k_c} = \begin{cases} 
2.4(1-C_k)^2 + 0.1 \delta^*_{R E k} & 0.75 < C_k < 1.25 \\
[1-C_k] \delta^*_{R E k} & 0 < C_k \leq 0.75 \text{ or } C_k \geq 1.25 \end{cases}
$$

(4)

$C_k$ [6] method is equivalent to the GCI, but with a variable $FS$ that increases linearly with the distance of solutions increases from the AR, as shown in Fig. 1.

Verification studies have shown that the estimates of $U_k$ and $U_{k_c}$ using Eqns. (3) and (4) are too conservative for $p_k > p_{k_o}$, as explained previously. An improved approach is to reflect the uncertainty itself with respect to the distance from the AR instead of reflecting the $FS$. First, $r^{p_k}_k$ in Eqn. (1) is re-expressed based on the definition of $C_k$:

$$
r^{p_k}_k = C_k \left( r^{p_{k_o}}_k - 1 \right) + 1
$$

(5)

Second, Eqn. (1) is substituted into Eqn. (3) for $0 < C_k \leq 0.875$ with the use of Eqn. (5), which results in an alternative form of $U_k$

$$
U_k = \left[ 2(1-C_k) + 1 \right] \frac{\delta^*_{R E k}}{C_k \left( r^{p_{k_o}}_k - 1 \right)} \quad 0 < C_k \leq 0.875
$$

(6)
Third, to enforce the same $U_k$ for $1.125 \leq C_k < 2$ as the $U_k$ at the same distance to $C_k = 1$ within $0 < C_k < 0.875$, $C_k$ in Eqn. (6) is replaced by $2 - C_k$. Thus for the same $r_k$, $p_{k1}$, and $\varepsilon_{k1}$, Equation (6) becomes:

$$
U_k = \left\{ \frac{C_k}{2 - C_k} \left[ 2(C_k - 1) + 1 \right] \right\} \delta^{\ast}_{RE_k} \quad 1.125 \leq C_k < 2 \quad (7)
$$

Additionally, two 3rd order polynomials instead of two quadratic functions as used in [6] are used to generate smoother curves of $FS$ for $0.875 < C_k \leq 1.0$ and $1.0 < C_k < 1.125$, respectively. The use of higher order polynomials allows not only the $FS$ magnitude but also the first order derivative of $FS$ with respect to $C_k$ to be continuous at $C_k = 0.875$, 1.0, and 1.125, which are used to determine the four unknown coefficients for each polynomial. Following [6], the magnitude and 1st order derivative of $FS$ at $C_k = 1$ are 1.1 and 0, respectively. Incorporating all of the above revisions the $U_k$ and $U_{kc}$ are given by:
Compared to $C_k$ [6], the improved $C_k$ method introduces an additional term $C_k/(2-C_k)$ to compute $U_k$ for $1.125 \leq C_k < 2$. When $C_k$ increases, this term increases rapidly from 1 to infinity, which amplifies $FS$ when solutions are further away from the AR. The improved $C_k$ method is only applicable for $0 < C_k < 2$. $C_k=0$ is the border of convergence and divergence such that grid errors/uncertainties are infinite due to infinite $\delta_{RE_h}$ as a result of $p_k=0$, i.e. solution changes for the medium and fine grids are equal to those for the coarse and medium grids. For $C_k > 2$, solutions are too far from the AR and also regarded as divergent. Figure 1 compares $FS$ predicted by the improved $C_k$, $C_k$ [6], and GCI methods with a zoomed in view near the AR shown in Fig. 2.

Compared with the $C_k$ [6] method, the improved $C_k$ method is more conservative for $C_k > 0.875$ except with the same $FS$ at $C_k=1$. The intersection points between the improved $C_k$ and GCI methods depends on value $FS$ used in GCI, e.g., for $FS=1.25$ intersection points are $C_k= (0.875, 1.06)$ and $(0.75, 1.12)$ for uncorrected and corrected solutions, respectively. When solutions are between the intersection points, i.e., closer to the AR, the GCI method is more conservative than the improved $C_k$ method. When
solutions are outside the intersection points, i.e., further from the AR, the GCI method is less conservative than the improved $C_k$ method.

![Figure 2](image)

**Figure 2.** A zoomed in view of factors of safety for correction factor and GCI verification methods.

### III. EXAMPLES FOR SHIP HYDRODYNAMICS APPLICATIONS

To demonstrate that the improved $C_k$ method predicts more reasonable intervals of grid uncertainties than $C_k$ [6] and GCI methods for industrial applications, the three methods are applied for a recent study [7] that used computational towing tank procedures for single run curves of resistance and propulsion for the high-speed transom ship Athena barehull with a skeg using the general-purpose solver CFDSHIP-Iowa-V.4 [8]. Extensive verification (for grid) and validation (not shown herein) studies are conducted by continuously refining the grid from the coarsest grid (grid 7 with 360,528 points; $y^+_1 = 4.26$) to the finest grid (grid 1 with 8.1 million points; $y^+_1 = 1.52$) for the Athena bare hull with skeg with 2 degrees of freedom (pitch and heave) at Froude number (Fr) 0.48. The grids are designed with a systematic grid refinement ratio $r_G = 2^{1/4}$, which allows 9 sets of grids for verification and validation (V&V) with 5 sets with $r_G =$
$2^{1/4}$ (5,6,7; 4,5,6; 3,4,5; 2,3,4; and 1,2,3), 3 sets with $r_G = 2^{1/2}$ (3,5,7; 2,4,6; and 1,3,5), and 1 set with $r_G = 2^{3/4}$ (1,4,7). Figure 3(a) and 3(c) show the solutions with EFD data for the resistance coefficients and ship motions, respectively. Figure 3(b) and 3(d) show the relative solution changes between two successive grids with iterative errors for the resistance coefficients and ship motions, respectively. The coarsest grid 7 is too coarse as its solution is out of the trend shown for the other grid solutions. $\varepsilon_N$ shows systematic decreasing for $C_{TX}$, $C_{F_X}$, and trim with $U_I < \varepsilon_N$ for the coarse grids. $\varepsilon_N$ shows oscillatory decreasing for $C_{P_X}$ and sinkage, which is caused by the problem of separating the iterative errors $U_I$ and $\varepsilon_N$ for the fine grids as they are of the same order of magnitude. Overall $U_I$ is insensitive to the refinement of grids and the average $U_I$ is 0.18%, 0.15%, and 0.2% for $C_{TX}$, sinkage and trim, respectively. $C_{TX}$ monotonically converges for 6 sets of grids except those with the coarsest grid 7 involved, whereas motions are more difficult to converge. Verification results for monotonic converged solutions are presented in Tables 1 and 2 for the total resistance coefficient and ship motions, respectively. $C_G$ shows large range of oscillations ($0.07 \leq C_G \leq 2.42$ for $C_{TX}$ and $0.40 \leq C_G \leq 16.92$ for ship motions) indicating that the solutions are not yet in the AR.

Table 1. Verification study for $C_{TX}$ of Athena bare hull with skeg ($Fr=0.48$). $U_G$ is $\%S_{fine}$; $C_{TX}$ is based on static wetted area; Factor of safety for GCI is 1.25

<table>
<thead>
<tr>
<th>Grids</th>
<th>Refinement Ratio</th>
<th>$R_G$</th>
<th>$p_G$</th>
<th>$C_G$</th>
<th>$U_G$ (%)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>($\varepsilon_G$, $\varepsilon_G$)</td>
<td></td>
<td></td>
<td></td>
<td>$Xing$</td>
</tr>
<tr>
<td>2, 4, 6</td>
<td>$\sqrt{2}$</td>
<td>0.63</td>
<td>1.32</td>
<td>0.58</td>
<td>4.90</td>
</tr>
<tr>
<td>1, 3, 5</td>
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<td>0.40</td>
<td>2.66</td>
<td>1.51</td>
<td>3.59</td>
</tr>
<tr>
<td>4, 5, 6</td>
<td>$\sqrt{2}$</td>
<td>0.97</td>
<td>0.16</td>
<td>0.07</td>
<td>125.2</td>
</tr>
<tr>
<td>3, 4, 5</td>
<td>$\sqrt{2}$</td>
<td>0.80</td>
<td>1.27</td>
<td>0.59</td>
<td>7.23</td>
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<tr>
<td>2, 3, 4</td>
<td>$\sqrt{2}$</td>
<td>0.60</td>
<td>2.98</td>
<td>1.64</td>
<td>8.73</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>$\sqrt{2}$</td>
<td>0.50</td>
<td>4.00</td>
<td>2.42</td>
<td>—</td>
</tr>
</tbody>
</table>

As shown in Table 1, $U_G$ of $C_{TX}$ for grids (4,5,6) is unreasonable large as it is too far away from the asymptotic range. $U_G$ of $C_{TX}$ for grids (1,2,3) using the $C_k$ [6] is
unreasonable small due to the deficiency discussed above. Excluding these two numbers, the average $U_G$ are 2.53%, 4.39%, and 6.11% for the GCI, $C_k$ [6], and improved $C_k$ methods, respectively. For $p_G > p_{G_{id}}$ on grids (2,3,4), the improved $C_k$ method predicts more reasonable $U_G$ (8.73%), which is 2 times the magnitude using $C_k$ [6] method and one order of magnitude larger than that using GCI. When solutions are closer to the AR, the differences between the $U_G$ using the three methods decrease. As shown in Table 2 for trim on grids (1,3,5) where $C_G=1.09$, $U_G$ is of the same order of magnitude for the three methods. Nonetheless, the improved $C_k$ method is more conservative than GCI and GCI is more conservative than the $C_k$ [6] method, which is consistent with the observation in Fig. 2 for $1.06 < C_G \leq 1.125$.

Table 2. Verification study for motions of Athena bare hull with skeg ($Fr=0.48$). $U_G$ is $\%S_{\text{fine}}$; Factor of safety for GCI is 1.25

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Grids</th>
<th>Refinement Ratio</th>
<th>$R_G$</th>
<th>$p_G$</th>
<th>$C_G$</th>
<th>$U_G$ (%)</th>
</tr>
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<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>Xing and</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stern</td>
<td>$C_k$ [6]</td>
</tr>
<tr>
<td>Sinkage</td>
<td>1, 3, 5</td>
<td>$\sqrt[4]{2}$</td>
<td>0.31</td>
<td>3.4</td>
<td>2.25</td>
<td>1.80</td>
</tr>
<tr>
<td>Sinkage</td>
<td>2, 3, 4</td>
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<td>12</td>
<td>16.92</td>
<td>—</td>
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<tr>
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<td>$\sqrt[4]{2}$</td>
<td>0.48</td>
<td>2.13</td>
<td>1.09</td>
<td>$4.67$</td>
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<tr>
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<td>0.89</td>
<td>0.40</td>
<td>42.87</td>
</tr>
<tr>
<td>Trim</td>
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<td>0.53</td>
<td>3.69</td>
<td>2.16</td>
<td>—</td>
</tr>
<tr>
<td>Trim</td>
<td>1, 2, 3</td>
<td>$\sqrt[4]{2}$</td>
<td>0.53</td>
<td>3.71</td>
<td>2.18</td>
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</tr>
</tbody>
</table>

Overall the improved $C_k$ method provides more reasonable uncertainty estimates for $p_G > p_{G_{id}}$ than the $C_k$ [6] and GCI methods. More accurate and efficient iterative methods (e.g. multi-grid) are needed to speed up the convergence and reduce the $U_l$ especially for the fine grids for improved assessment of grid convergence. Further refinement with $y_i^+ < 1$ may also help reach the AR but requires at least 38 million grid points, which raises issues of code efficiency and available computer resources.
Figure 3. Verification for resistance and motions for Athena bare hull with skeg (Fr=0.48): (a) resistance coefficients, (b) relative change \( \varepsilon_{N} = \|S_{N-1} - S_{N}\| / S_{N} \times 100 \) and iterative errors for resistance coefficients, (c) sinkage and trim, (d) relative change \( \varepsilon_{N} \) and iterative errors for sinkage and trim.

IV. CONCLUSIONS AND FUTURE WORK

Improved factors of safety for quantitative estimates for grid and time convergence uncertainties for CFD solutions are proposed for situations when Richardson extrapolation estimated order of accuracy \( p_k \) is larger than the theoretical order of
accuracy $p_{k\theta}$. The improved uncertainty estimates are shown to provide more reasonable intervals of uncertainty for $p_k > p_{k\theta}$ and $1<C_k<2$.

Since numerical solutions of the analytical benchmarks conducted so far approach the AR with $p_k < p_{k\theta}$, it is desirable to validate current verification procedures using more advanced numerical benchmarks for complex flows such as backward-facing step flow as the solutions will likely approach the AR similarly as industrial applications, i.e., with oscillatory $1-C_k$. The more advanced numerical benchmarks can also be used to validate the validation procedures.

References


