Handout: Channel Flow - Definitions and Governing Equations

Definition

Fig. 1 Sketch of channel flow (Pope’s book)

Assumption:

\[ L \gg \delta \]

\[ b \gg \delta \]

We focus on the fully developed region (large \( x \)) in which velocity statistics no longer vary with \( x \). As the duct has also a large aspect ratio, the turbulence statistics are homogeneous in \( x \) and \( z \) (away from the lateral walls).

The lower and upper walls are situated at \( y=0 \) and \( y=2\delta \) with the channel centerline situated at \( y=\delta \).

Reynolds numbers:

\[ \text{Re} = \frac{2\delta \bar{U}}{v} ; \quad \bar{U} \text{ is bulk velocity} \]

\[ \text{Re}_0 = \frac{\delta U_0}{v} ; \quad U_o = \langle U \rangle_{y=\delta} \text{ is the centerline velocity,} \]

\(<\>\text{ denotes Reynolds averaged quantity}\)

Laminar flow for \( \text{Re}<1350 \), Fully developed for \( \text{Re} > 1800 \), transitional effects present up to \( \text{Re}=3000 \).
Governing Equations

Continuity

\[
\frac{\partial \langle U \rangle}{\partial x} + \frac{\partial \langle V \rangle}{\partial y} = 0
\]

\[
\Rightarrow \frac{\partial \langle V \rangle}{\partial y} = 0
\]

Integrate in y with \( \langle V \rangle_{y=0} = 0 \)

\[
\Rightarrow \langle V \rangle = 0
\]

Lateral Momentum Equation

\[
\langle U \rangle \frac{\partial \langle V \rangle}{\partial x} + \langle V \rangle \frac{\partial \langle V \rangle}{\partial y} = -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial y} + \nu \frac{\partial^2 \langle V \rangle}{\partial x^2} + \nu \frac{\partial^2 \langle V \rangle}{\partial y^2} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y}
\]

\[
\Rightarrow \frac{\partial}{\partial y} \left( \frac{\langle p \rangle}{\rho} + \langle v^2 \rangle \right) = 0
\]

Integrate from 0 to y

\[
\langle v^2 \rangle + \frac{\langle p \rangle}{\rho} = \frac{p_w(x)}{\rho}
\]

Differentiate with respect to x

\[
\frac{\partial \langle p \rangle}{\partial x} = \frac{\partial p_w}{\partial x}
\]

\[
\Rightarrow \quad \text{Pressure gradient uniform across flow}
\]

Axial Momentum Equation

\[
\langle U \rangle \frac{\partial \langle U \rangle}{\partial x} + \langle V \rangle \frac{\partial \langle U \rangle}{\partial y} = -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x} + \nu \frac{\partial^2 \langle U \rangle}{\partial x^2} + \nu \frac{\partial^2 \langle U \rangle}{\partial y^2} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y}
\]

Total shear stress (viscous + Reynolds stress):

\[
\tau = \rho v \frac{\partial \langle U \rangle}{\partial y} - \rho \langle uv \rangle
\]
\[ \Rightarrow \frac{\partial \tau}{\partial y} = \frac{\partial p_w}{\partial x} \]

⇒ Total shear stress compensated by pressure gradient

\[ \frac{\partial \tau}{\partial y} \neq f(x) \text{ and } \frac{\partial p_w}{\partial y} \neq f(y) \]

\[ \Rightarrow \frac{\partial \tau}{\partial y} = \frac{\partial p_w}{\partial x} = \text{const} \]

\[ \tau(y) = \tau_w \left( 1 - \frac{y}{\delta} \right) \]

Skin friction coefficients

\[ c_f \equiv \frac{\tau_w}{1/2 \rho U_0^2} \]

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Fig. 7.3. Profiles of the viscous shear stress, and the Reynolds shear stress in turbulent channel flow: DNS data of Kim et al. (1987): dashed line, Re = 5,600; solid line, Re = 13,750.

Figure 7.3 from Pope's Turbulent Flow book: Reynolds stress, viscous stress, and total stress (DNS of KIM, Mion, Moser (1987))
Wall Shear Stress

$\tau_w$ given only by viscous stress as $\langle uv \rangle = 0$ at the wall

$$\tau_w = \rho \nu \frac{\partial \langle U \rangle}{\partial y} \bigg|_{y=0}$$

Friction velocity:

$$u_\tau \equiv \sqrt{\frac{\tau_w}{\rho}}$$

Viscous length scale:

$$\delta_v \equiv \frac{\nu}{u_\tau} = \frac{v}{u_\tau} \sqrt{\frac{\rho}{\tau_w}}$$

$$\Rightarrow \text{Re}_v \equiv \frac{\delta_1 u_\tau}{v} = 1$$

Friction Reynolds number (Reynolds number defined with the friction velocity):

$$\text{Re}_f = \frac{u_\tau \delta}{v} = \frac{\delta}{\delta_v}$$

Wall units:

$$y^+ = \frac{y}{\delta_v} = \frac{u_\tau y}{v} \quad \text{(similar to local Reynolds number)}$$

Fig. 7.4. Profiles of the fractional contributions of the viscous and Reynolds stresses to the total stress. DNS data of Kim et al. (1987): dashed lines, $Re = 5600$; solid lines, $Re = 13750$.

- Outer layer: $y^+ > 50$ (viscous stresses can be neglected, direct effect of viscosity is negligible)
• Viscous wall region: \( y^+ < 50 \)

• Viscous sublayer: \( y^+ < 5 \) (Reynolds stresses are negligible)

As \( \text{Re} \) increases, the fraction of the channel occupied by the viscous wall region decreases, since \( \frac{\delta_v}{\delta} \approx \text{Re}^{-1} \)