Handout: Channel Flow - Mean Velocity Profiles

Similarity

\[ \langle U \rangle = f(\rho, \nu, \delta, y, \frac{dp_w}{dx}) = f(\rho, \nu, \delta, y, u_\tau) \]

\( \Rightarrow 6 - 3 = 3 \) non-dimensional groups

\[ \frac{\langle U \rangle}{u_\tau} = f\left(\frac{y}{\delta}, \frac{\delta}{\delta_v}\right) = f\left(\frac{y}{\delta}, \text{Re}_\tau\right) \]

One can do the same analysis for the Velocity Gradient:

\[ \frac{d\langle U \rangle}{dy} = \frac{u_\tau}{y} \Phi\left(\frac{y}{\delta}, \frac{y}{\delta_v}\right) \]

\( \delta_v \) is the appropriate length scale in the viscous wall region, while \( \delta \) is the appropriate length scale in the outer layer

The law of the wall

Prandtl (1925): In the inner layer (\( \frac{y}{\delta} < 1 \)) and at high Reynolds numbers \( < U > \) is determined only by the viscous scales

\[ \Rightarrow \frac{d\langle U \rangle}{dy} = \frac{u_\tau}{y} \Phi\left(\frac{y}{\delta_v}\right) \quad \text{for} \quad \frac{y}{\delta} < 1 \quad (1) \]

with

\[ u^+ = \frac{\langle U \rangle}{u_\tau} \quad \text{and} \quad y^+ = \frac{y}{\delta_v} \]

\[ \Rightarrow \frac{du^+}{dy^+} = \frac{1}{y^+} \Phi\left(y^+\right) \quad (2) \]

The integral of the previous equation is the Law of the Wall valid in viscous sublayer and the inner layer \( (y/d < 0.1) \)
\[ u^+ = f_w(y^+) \]  

where

\[ f_w(y^+) = \frac{1}{y^+} \int_0^{y^+} \Phi_f(y^+) dy^+ \]

**Viscous Sublayer**

From equation (3)

\[ f_w = u^+ = \frac{\langle U \rangle}{u_\tau} \]

\[ \Rightarrow f_w(0) = 0 \]

\[ f_w'(0) = \left. \frac{du^+}{dy^+} \right|_{y=0} = \frac{\delta_v \langle U \rangle}{u_\tau \partial y^+} \bigg|_{y=0} \]

\[ = \frac{\nu}{u_\tau} \frac{\tau_w}{\nu} = \frac{\rho}{\tau_w} \frac{\tau_w}{\rho} = 1 \]

Hence, the Taylor series expansion for small \( y^+ \)

\[ u^+ = f_w(y^+) = f_w(0) + f_w'(0)y^+ + O(y^{+2}) \]

\[ \Rightarrow u^+ = y^+ \]
The Log Law

At large Reynolds numbers, the outer part of the inner layer corresponds to large \( y^+ \) where viscosity has little effect (\( \Phi \) independent of \( \frac{y}{\delta_v} \))

\[
\Phi = \Phi\left(\frac{y}{\delta_v}, \frac{\delta}{\delta_v}\right)
\]

\( y^+ \gg 1 \Rightarrow \frac{y}{\delta_v} = \frac{y}{\delta} \gg 1 \)

\[
\Rightarrow \frac{y}{\delta} \gg \frac{\delta_v}{\delta} = \frac{v}{u_\tau \delta} = \text{Re}_\tau^{-1}
\]

Also \( \Phi \) independent of \( \frac{y}{\delta} \) for

\[
\frac{y}{\delta} \ll 1 \quad \text{let's say} \quad \frac{y}{\delta} < 0.1
\]

\[
\Rightarrow \frac{y}{\delta} \frac{\delta_v}{\delta} < 0.1
\]

\[
\Rightarrow y^+ < 0.1 \frac{\delta}{\delta_v} = 0.1 \frac{\delta u_\tau}{\nu} = 0.1 \text{Re}_\tau
\]

Fig. 7.5. Near-wall profiles of mean velocity from the DNS data of Kim et al. (1987): dashed line, Re = 5,600; solid line, Re = 13,750; dot-dashed line, \( u^+ = y^+ \).
where the edge of the inner layer was defined at $0.1\delta$.

So if $Re_\tau$ is such that $\frac{y}{\delta} \gg Re_\tau^{-1}$ and $y^+ = y/\delta_y < 0.1Re_\tau$

$\Phi_I(y^+)$ independent of both $\frac{y}{\delta_y}$ (viscosity) and of $\frac{y}{\delta}$, thus from equation (2):

$$\Rightarrow \Phi_I = \text{const} = \frac{1}{\kappa}$$

$$\Rightarrow \frac{du^+}{dy^+} = \frac{1}{\kappa y^+}$$

$$\Rightarrow u^+ = \frac{1}{\kappa} \ln y^+ + B$$

with

$$\kappa = 0.41 \quad \text{(Von Karman constant)}$$

$$B = 5.2$$

*Fig. 7.7. Mean velocity profiles in fully developed turbulent channel flow measured by Wei and Willmarth (1989); \(\circ\), $Re_0 = 2,970$; \(\square\), $Re_0 = 14,914$; \(\bigtriangleup\), $Re_0 = 22,776$; \(\triangledown\), $Re_0 = 39,582$; line, the log law, Eqs. (7.43)–(7.44).*
Fig. 7.8. A sketch showing the various wall regions and layers defined in terms of $y^+ = y/\delta$, and $y/\delta$, for turbulent channel flow at high Reynolds number ($Re_t = 10^4$).

Fig. 7.13. Regions and layers in turbulent channel flow as functions of the Reynolds number.
Velocity Defect Law

In the outer layer \((y^+ > 50)\) the assumption that \(\Phi\) is independent of \(\nu\) implies that, for large \(\frac{y}{\delta_v}\), \(\Phi\) tends asymptotically to a function of \(y/\delta\) only:

\[
\frac{d\langle U\rangle}{dy} = \frac{u_\tau}{y} \Phi_o \left( \frac{y}{\delta} \right) \tag{4}
\]

By integrating in \(y\) from \(y\) to \(\delta\) (channel centerline)

\[
\int_y^\delta \frac{d\langle U\rangle}{dy} dy = \int_y^\delta \frac{u_\tau}{y} \Phi_o \left( \frac{y'}{\delta} \right) dy' = \frac{U_o - \langle U\rangle}{u_\tau} = F_D \left( \frac{y}{\delta} \right) \quad \text{with} \quad F_D \left( \frac{y}{\delta} \right) = \int_y^{\delta y} \Phi_o (y') dy' \tag{5}
\]

Unlike the law of the wall function \(f_w(y^+)\) which is universal, the function \(F_D \left( \frac{y}{\delta} \right)\) is different in different flows.

At sufficiently high Reynolds numbers (approximately \(Re>20,000\)) there is an overlap region between the inner layer and the outer layer (see Figs. 7.8 and 7.13). In this region both equations (1) and (4) are valid

\[
\frac{y}{u_\tau} \frac{d\langle U\rangle}{dy} = \Phi \left( \frac{y}{\delta_v} \right) = \Phi_o \left( \frac{y}{\delta} \right) \quad \text{for} \quad \delta_v \ll y \ll \delta
\]

This can be satisfied only if the two functions on the right hand side are equal to the same constant \((1/\kappa)\), which leads to

\[
\Rightarrow \frac{d\langle U\rangle}{dy} = \frac{u_\tau}{y\kappa} \quad \text{for} \quad \delta_v \ll y \ll \delta
\]

For small \(y/\delta\), the velocity defect law (equation 5) can be written as:

\[
\frac{U_0 - \langle U\rangle}{u_\tau} = F_D \left( \frac{y}{\delta} \right) = -\frac{1}{\kappa} \ln \frac{y}{\delta} + B_1 \tag{6}
\]
The value of the flow-dependent constant $B_1$ is $\sim 0.2$ based on DNS and $\sim 0.7$ based on experimental data for channel flows.

Statistics in turbulent channel flows:

Table 7.2. Statistics in turbulent channel flow, obtained from the DNS data of Kim et al. (1987), $Re = 13,750$

<table>
<thead>
<tr>
<th>Location</th>
<th>Peak production $y^* = 11.8$</th>
<th>Log law $y^* = 98$</th>
<th>Centerline $y^* = 395$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle u^2 \rangle / k$</td>
<td>1.70</td>
<td>1.02</td>
<td>0.84</td>
</tr>
<tr>
<td>$\langle w^2 \rangle / k$</td>
<td>0.04</td>
<td>0.39</td>
<td>0.57</td>
</tr>
<tr>
<td>$\langle w^2 \rangle / k$</td>
<td>0.26</td>
<td>0.59</td>
<td>0.59</td>
</tr>
<tr>
<td>$\langle uw \rangle / k$</td>
<td>$-0.116$</td>
<td>$-0.285$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\rho_{ui}$</td>
<td>$-0.44$</td>
<td>$-0.45$</td>
<td>$0$</td>
</tr>
<tr>
<td>$Sk/s$</td>
<td>15.6</td>
<td>3.2</td>
<td>0</td>
</tr>
<tr>
<td>$P/e$</td>
<td>1.81</td>
<td>0.91</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 7.14. Reynolds stresses and kinetic energy normalized by the friction velocity against $y^*$ from DNS of channel flow at $Re = 13,750$ (Kim et al. 1987).
Fig. 7.15. Profiles of Reynolds stresses normalized by the turbulent kinetic energy from DNS of channel flow at $Re = 13,750$ (Kim et al. 1987).

Fig. 7.16. Profiles of the ratio of production to dissipation ($P/\varepsilon$), normalized mean shear rate ($Sk/\varepsilon$), and shear stress correlation coefficient ($\rho_{uv}$) from DNS of channel flow at $Re = 13,750$ (Kim et al. 1987).
Fig. 7.18. The turbulent-kinetic-energy budget in the viscous wall region of channel flow: terms in Eq. (7.64) normalized by viscous scales. From the DNS data of Kim et al. (1987). $Re = 13,750$. 