Summary of Experimental Uncertainty Assessment Methodology with Example

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Introduction

- Experiments are an essential and integral tool for engineering and science

- Uncertainty estimates are imperative for risk assessments in design both when using data directly or in calibrating and/or validating simulations methods

- True values are seldom known and experiments have errors due to instruments, data acquisition, data reduction, and environmental effects

- Determination of truth requires estimates for experimental errors, i.e., uncertainties
Introduction

- Uncertainty analysis (UA): rigorous methodology for uncertainty assessment using statistical and engineering concepts

- ASME and AIAA standards (e.g., ASME, 1998; AIAA, 1995) are the most recent updates of UA methodologies, which are internationally recognized

- Presentation purpose: to provide summary of EFD UA methodology accessible and suitable for student and faculty use both in classroom and research laboratories
Terminology

- **Accuracy**: closeness of agreement between measured and true value

- **Error**: difference between measured and true value

- **Uncertainties** ($U$): estimate of errors in measurements of individual variables $X_i$ ($U_{x_i}$) or results ($U_r$)

- Estimates of $U$ made at 95% confidence level, on large data samples (at least 10/measurement)
Terminology

- **Bias error** ($\beta$): fixed, systematic
- **Bias limit** ($B$): estimate of $\beta$
- **Precision error** ($\varepsilon$): random
- **Precision limit** ($P$): estimate of $\varepsilon$
- **Total error**: $\delta = \beta + \varepsilon$
Terminology

- **Measurement systems** for individual variables $X_i$: instrumentation, data acquisition and reduction procedures, and operational environment (laboratory, large-scale facility, in situ)

- Results expressed through **data-reduction equations (DRE)**
  \[ r = r(X_1, X_2, X_3, \ldots, X_j) \]

- Estimates of errors are meaningful only when considered in the **context of the process** leading to the value of the quantity under consideration

- Identification and quantification of **error sources** require considerations of:
  - steps used in the process to obtain the measurement of the quantity
  - the environment in which the steps were accomplished
**Terminology**

- **Block diagram:** elemental error sources, individual measurement systems, measurement of individual variables, data reduction equations, and experimental results

\[
\begin{align*}
  r &= r (X_1, X_2, \ldots, X_J) \\
  B_r &= B_r, P_r
\end{align*}
\]
Uncertainty propagation equation

- One variable, one measurement

\[ \delta_r = r(X_i) - r_{true}(X_i) = \delta X_i \frac{dr}{dX_i} \]
Uncertainty propagation equation

- Two variables, the $k$th set of measurements $(x_k, y_k)$

\[ r = r(x, y) \]

\[ x_k = x_{true} + \beta_{x_k} + \varepsilon_{x_k} \]

\[ y_k = y_{true} + \beta_{y_k} + \varepsilon_{y_k} \]

\[ r_k - r_{true} = \frac{\partial r}{\partial x} (x_k - x_{true}) + \frac{\partial r}{\partial y} (y_k - y_{true}) + R_2 \]

The total error in the $k$th determination of $r$

\[ \delta_{r_k} = r_k - r_{true} = \theta_x (\beta_{x_k} + \varepsilon_{x_k}) + \theta_y (\beta_{y_k} + \varepsilon_{y_k}) \] (1)
Uncertainty propagation equation

- A measure of \( \delta_r \) is

\[
\sigma^2_{\delta_r} = \lim_{N \to \infty} \left[ \frac{1}{N} \sum_{k=1}^{N} (\delta_{r_k})^2 \right]
\]  

Substituting (2) in (1), and assuming that bias/precision errors are correlated

\[
\sigma^2_{\delta_r} = \theta_x^2 \sigma^2_{\beta_x} + \theta_y^2 \sigma^2_{\beta_y} + 2\theta_x \theta_y \sigma_{\beta_x \beta_y} + \theta_x^2 \sigma^2_{\epsilon_x} + \theta_y^2 \sigma^2_{\epsilon_y} + 2\theta_x \theta_y \sigma_{\epsilon_x \epsilon_y}
\]  

\( \sigma \)'s are not known; use estimates for the variances and covariances of the distributions of the total, bias, and precision errors

\[
u^2_c = \theta_x^2 b_x^2 + \theta_y^2 b_y^2 + 2\theta_x \theta_y b_{xy} + \theta_x^2 S_x^2 + \theta_y^2 S_y^2 + 2\theta_x \theta_y S_{xy}
\]

The total uncertainty of the results at a specified level of confidence is

\[U_r = Ku_c\]

\((K = 2\) for 95% confidence level)
Uncertainty propagation equation

- Generalizing (3) for \( J \) variables

\[
U_r^2 = \sum_{i=1}^{J} \theta_i^2 B_i^2 + 2 \sum_{i=1}^{J-1} \theta_i \theta_k B_{ik} + \sum_{i=1}^{J} \theta_i^2 P_i^2 + 2 \sum_{i=1}^{J-1} \theta_i \theta_k P_{ik}
\]

\[
\theta_i = \frac{\partial r}{\partial X_i}
\]

sensitivity coefficients

Example:

\[
C_D = \frac{D}{\frac{1}{2} \rho U^2 A} = C_D(D, \rho, U, A)
\]

\[
U_{C_D}^2 = \sum_{i=1}^{J} \theta_i^2 B_i^2 + \sum_{i=1}^{J} \theta_i^2 P_i^2
\]

\[
= \left( \frac{\partial C_D}{\partial D} \right)^2 (B_D^2 + P_D^2) + \left( \frac{\partial C_D}{\partial \rho} \right)^2 (B_\rho^2 + P_\rho^2) + \left( \frac{\partial C_D}{\partial U} \right)^2 (B_U^2 + P_U^2) + \left( \frac{\partial C_D}{\partial A} \right)^2 (B_A^2 + P_A^2)
\]
Single and multiple tests

■ **Single test:** one set of measurements \((X_1, X_2, \ldots, X_j)\) for \(r\)

■ **Multiple tests:** many sets of measurements \((X_1, X_2, \ldots, X_j)\) for \(r\)

■ **The total uncertainty of the result** (single and multiple)

\[
U_r^2 = B_r^2 + P_r^2
\]  \hspace{1cm} (4)

■ **\(B_r\):** determined in the same manner for single and multiple tests

■ **\(P_r\):** determined differently for single and multiple tests
Bias limits (single and multiple tests)

- $B_r$ given by:
  \[ B_r^2 = \sum_{i=1}^{J} \theta_i^2 B_i^2 + 2 \sum_{i=1}^{J-1} \sum_{k=i+1}^{J} \theta_i \theta_k B_{ik} \]

- Sensitivity coefficients
  \[ \theta_i = \frac{\partial r}{\partial X_i} \]

- $B_i$: estimate of calibration, data acquisition, data reduction, and conceptual bias errors for $X_i$

- $B_{ik}$: estimate of correlated bias limits for $X_i$ and $X_k$
  \[ B_{ik} = \sum_{\alpha=1}^{L} (B_i)_\alpha (B_k)_\alpha \]
**Precision limits (multiple tests)**

- **Precision limit of the result (end to end):**

\[
P_r = \frac{tS_r}{\sqrt{M}}
\]

\(t\): coverage factor (\(t = 2\) for \(N > 10\))

\(S_r\): standard deviation for \(M\) readings of the result

- **The average result:**

\[
\bar{r} = \frac{1}{M} \sum_{k=1}^{M} r_k
\]
Precision limits (single test)

- Precision limit of the result (end to end):

\[ P_r = t S_r \]

- \( t \): coverage factor \((t = 2 \text{ for } N > 10)\)
- \( S_r \): the standard deviation for the \( N \) readings of the result. It is not available for single test. Use of “best available information” (literature, inter-laboratory comparison, etc.) needed.
EFD Validation

- Conduct uncertainty analysis for the results:
  - EFD result: \( A \pm U_A \)
  - Benchmark or EFD data: \( B \pm U_B \)

\[
E = B - A
\]

\[
U_E^2 = U_A^2 + U_B^2
\]

- Validation:
  \[
|E| < U_E
\]
Recommendations for implementation

- Determine data reduction equation: \( r = r(X_1, X_2, \ldots, X_j) \)
- Construct the block diagram
- Identify and estimate sources of errors
- Establish relative significance of the bias limits for the individual variables
- Estimate precision limits (end-to-end procedure recommended)
- Calculate total uncertainty using equation (4)
- Report total error, bias and precision limits for the final result
Recommendations for implementation

- Recognition of the uncertainty analysis (UA) importance
- Full integration of UA into all phases of the testing process
- Simplified UA:
  - dominant error sources only
  - use of previous data
  - end-to-end calibration and estimation of errors
- Full documentation:
  - Test design, measurement systems, data-stream in block diagrams
  - Equipment and procedure
  - Error sources considered
  - Estimates for bias and precision limits and estimating procedures
  - Detailed UA methodology and actual data uncertainty estimates
Experimental Uncertainty Assessment Methodology: Example for Measurement of Density and Kinematic Viscosity
A sphere of diameter $D$ falls a distance $\lambda$ at terminal velocity $V$ (fall time $t$) through a cylinder filled with 99.7% aqueous glycerin solution of density $\rho$, viscosity $\mu$, and kinematic viscosity $\nu (= \mu/\rho)$.

Flow situations:
- $Re = VD/\nu << 1$ (Stokes law)
- $Re > 1$ (asymmetric wake)
- $Re > 20$ (flow separates)
Test Design

- Assumption: $Re = \frac{VD}{\nu} << 1$

- Forces acting on the sphere:

  \[ W_a = F_g - F_b = F_d \]

  - Apparent weight

    \[ W_a = \gamma \nabla (S - 1) \]

    \[ \gamma = \rho g; \quad \nabla = \pi D^3 / 6; \quad S = \frac{\rho_{sphere}}{\rho} \]

  - Drag force (Stokes law)

    \[ F_d = 3\pi \mu VD \]
Test design

- Terminal velocity:
  \[ V = \frac{gD^2}{18 \nu} (S - 1); \quad V = \frac{\lambda}{t} \]

- Solving for \( \nu \) and substituting \( \lambda/t \) for \( V \)
  \[ \nu = \nu(D, t, \lambda, \rho) = \frac{gD^2 t}{18 \lambda} (S - 1) \]  
  \[ (5) \]

- Evaluating \( \nu \) for two different spheres (e.g., teflon and steel) and solving for \( \rho \)
  \[ \rho = \rho(D_t, t_t, D_s, t_s) = \frac{D_t^2 t_t \rho_t - D_s^2 t_s \rho_s}{D_t^2 t_t - D_s^2 t_s} \]  
  \[ (6) \]

- Equations (5) and (6): data reduction equations for \( \nu \) and \( \rho \) in terms of measurements of the individual variables: \( D_t, D_s, t_t, t_s, \lambda \)
Measurement Systems and Procedures

- Individual measurement systems:
  - $D_t$ and $D_s$ – micrometer; resolution 0.01mm
  - $\lambda$ – scale; resolution 1/16 inch
  - $t_t$ and $t_s$ - stopwatch; last significant digit 0.01 sec.
  - $T$ (temperature) – digital thermometer; last significant digit 0.1° F

- Data acquisition procedure:
  1. measure $T$ and $\lambda$
  2. measure diameters $D_t$ and fall times $t_t$ for 10 teflon spheres
  3. measure diameters $D_s$ and fall times $t_s$ for 10 steel spheres

- Data reduction is done at steps (5) and (6) by substituting the measurements for each test into the data reduction equation (6) for evaluation of $\rho$ and then along with this result into the data reduction equation (5) for evaluation of $\nu$
Block-diagram

**EXPERIMENTAL ERROR SOURCES**

- **SPHERE DIAMETER**
  - $X_D$
  - $B_D, P_D$

- **FALL DISTANCE**
  - $X_\lambda$
  - $B_\lambda, P_\lambda$

- **FALL TIME**
  - $X_t$
  - $B_t, P_t$

**INDIVIDUAL MEASUREMENT SYSTEMS**

**MEASUREMENT OF INDIVIDUAL VARIABLES**

**DATA REDUCTION EQUATIONS**

\[
\rho = \rho (X_D, X_t) = \frac{D_s^2 t_s \rho_s - D_t^2 t_t \rho_t}{D_s^2 t_s - D_t^2 t_t} / \frac{D^2 g(\frac{\rho_{sphere}}{\rho} - 1) t}{18 \lambda}
\]

\[
\nu = \nu (X_D, X_t, X_\rho, X_\lambda) = \frac{D^2 g(\frac{\rho_{sphere}}{\rho} - 1) t}{18 \lambda}
\]

**EXPERIMENTAL RESULTS**

- $\nu_{s,t}$
  - $B_{\nu_{s,t}}, P_{\nu_{s,t}}$
- $\rho$
  - $B_{\rho}, P_{\rho}$
Test results

Table 1. Gravity and sphere density constants

<table>
<thead>
<tr>
<th>Definitions</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravitational acceleration</td>
<td>$g$</td>
<td>9.81 m/s$^2$</td>
</tr>
<tr>
<td>Density of steel</td>
<td>$\rho_s$</td>
<td>7991 kg/m$^3$</td>
</tr>
<tr>
<td>Density of teflon</td>
<td>$\rho_t$</td>
<td>2148 kg/m$^3$</td>
</tr>
</tbody>
</table>

Table 2. Typical test results

<table>
<thead>
<tr>
<th>Trial</th>
<th>TEFLON</th>
<th>STEEL</th>
<th>RESULTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$D_t$</td>
<td>$t_t$</td>
<td>$D_s$</td>
</tr>
<tr>
<td></td>
<td>(m)</td>
<td>(sec)</td>
<td>(m)</td>
</tr>
<tr>
<td>1</td>
<td>0.00661</td>
<td>31.08</td>
<td>0.00359</td>
</tr>
<tr>
<td>2</td>
<td>0.00646</td>
<td>31.06</td>
<td>0.00358</td>
</tr>
<tr>
<td>3</td>
<td>0.00634</td>
<td>30.71</td>
<td>0.00359</td>
</tr>
<tr>
<td>4</td>
<td>0.00632</td>
<td>30.75</td>
<td>0.00359</td>
</tr>
<tr>
<td>5</td>
<td>0.00634</td>
<td>30.89</td>
<td>0.00359</td>
</tr>
<tr>
<td>6</td>
<td>0.00633</td>
<td>30.82</td>
<td>0.00359</td>
</tr>
<tr>
<td>7</td>
<td>0.00637</td>
<td>30.89</td>
<td>0.00359</td>
</tr>
<tr>
<td>8</td>
<td>0.00634</td>
<td>30.71</td>
<td>0.00359</td>
</tr>
<tr>
<td>9</td>
<td>0.00633</td>
<td>31.2</td>
<td>0.00359</td>
</tr>
<tr>
<td>10</td>
<td>0.00634</td>
<td>31.11</td>
<td>0.00359</td>
</tr>
<tr>
<td>Average</td>
<td>0.00637</td>
<td>30.91</td>
<td>0.00358</td>
</tr>
<tr>
<td>Std.Dev. ($S_t$)</td>
<td>$9.17\cdot10^{-5}$</td>
<td>0.18</td>
<td>$3.16\cdot10^{-6}$</td>
</tr>
</tbody>
</table>
Uncertainty assessment (multiple tests)

- **Density** $\rho$  
  \[
  \rho = \rho(D_t, t_t, D_s, t_s) = \frac{D_t^2 t_t \rho_f - D_s^2 t_s \rho_s}{D_t^2 t_t - D_s^2 t_s}
  \]

- **Bias limit**
  \[
  B^2_\rho = \theta^2_{D_t} B_{D_t}^2 + \theta^2_{t_t} B_{t_t}^2 + \theta^2_{D_s} B_{D_s}^2 + \theta^2_{t_s} B_{t_s}^2 + 2\theta_{D_t} \theta_{D_s} B_{D_t} B_{D_s} + 2\theta_{t_t} \theta_{t_s} B_{t_t} B_{t_s}
  \]

<table>
<thead>
<tr>
<th>Bias Limit</th>
<th>Magnitude</th>
<th>Percentage Values</th>
<th>Estimation</th>
</tr>
</thead>
</table>
| $B_{\rho} = B_{D_t} = B_{D_s}$ | 0.000005 m | 0.078% $D_t$  
0.14% $D_s$ | $\frac{1}{2}$ instrument resolution |
| $B_{t_t} = B_{t_s} = B_t$ | 0.01 s | 0.032% $t_t$  
0.083% $t_s$ | Last significant digit |

- **Sensitivity coefficients**:
  \[
  \theta_{D_t} = \frac{\partial \rho}{\partial D_t} = \frac{2 D_s^2 t_t t_s D_t (\rho_s - \rho_f)}{[D_t^2 t_t - D_s^2 t_s]^2} = 296,808 \text{ kg/m}^4
  \]

- **Precision limit**
  \[
  P_\rho = \frac{2 \cdot S_\rho}{\sqrt{M}}
  \]

- **Total uncertainty**
  \[
  U_\rho = \pm \sqrt{B^2_\rho + P^2_\rho}
  \]
### Uncertainty assessment (multiple tests)

#### Density $\rho$

<table>
<thead>
<tr>
<th>Term</th>
<th>Without correlated bias errors</th>
<th>With correlated bias errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Magnitude</td>
<td>% Values</td>
</tr>
<tr>
<td>$\theta_{D_i} B_{D_i}$</td>
<td>$1.48 \text{ kg/m}^3$</td>
<td>22.30% $B^2_\rho$</td>
</tr>
<tr>
<td>$\theta_{i_i} B_i$</td>
<td>$0.31 \text{ kg/m}^3$</td>
<td>0.95% $B^2_\rho$</td>
</tr>
<tr>
<td>$\theta_{D_i} B_{D_i}$</td>
<td>$-2.63 \text{ kg/m}^3$</td>
<td>70.60% $B^2_\rho$</td>
</tr>
<tr>
<td>$\theta_{i_i} B_i$</td>
<td>$-0.78 \text{ kg/m}^3$</td>
<td>6.15% $B^2_\rho$</td>
</tr>
<tr>
<td>$2\theta_{D_i} B_{D_i}, B_{D_i}^2$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$2\theta_{i_i}, B_i^2$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$B_\rho$</td>
<td>$3.13 \text{ kg/m}^3$</td>
<td>0.24% $\rho$</td>
</tr>
<tr>
<td>$P_{\rho}$</td>
<td>$16.91 \text{ kg/m}^3$</td>
<td>1.28% $\rho$</td>
</tr>
<tr>
<td>$U_{\rho}$</td>
<td>$17.20 \text{ kg/m}^3$</td>
<td>1.30% $\rho$</td>
</tr>
</tbody>
</table>
Uncertainty assessment (multiple tests)

- **Viscosity $\nu$** (DRE: $\nu = \nu(D, t, \lambda, \rho) = \frac{gD^2t}{18\lambda} (S - 1)$)

  - Calculations for teflon sphere
  - Bias limit
  - Precision limit
  - Total uncertainty

<table>
<thead>
<tr>
<th>Term</th>
<th>Magnitude</th>
<th>Percentage Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_\lambda$</td>
<td>$7.9\times10^{-4}$ m</td>
<td>0.13% $\lambda$</td>
</tr>
<tr>
<td>$\theta_DB_D$</td>
<td>$1.1\times10^{-6}$ m$^2$/s</td>
<td>5.97% $B^2_{\nu}$</td>
</tr>
<tr>
<td>$\theta_PB^2$</td>
<td>$4.27\times10^{-6}$ m$^2$/s</td>
<td>90.03% $B^2_{\nu}$</td>
</tr>
<tr>
<td>$\theta_tB_t$</td>
<td>$2.29\times10^{-7}$ m$^2$/s</td>
<td>0.26% $B^2_{\nu}$</td>
</tr>
<tr>
<td>$\theta_\lambda B_\lambda$</td>
<td>$-0.92\times10^{-6}$ m$^2$/s</td>
<td>3.74% $B^2_{\nu}$</td>
</tr>
<tr>
<td>$B_{\nu_i}$</td>
<td>$4.5\times10^{-6}$ m$^2$/s</td>
<td>0.64% $\nu_i$</td>
</tr>
<tr>
<td>$P_{\nu_i}$</td>
<td>$1.01\times10^{-5}$ m$^2$/s</td>
<td>1.43% $\nu_i$</td>
</tr>
<tr>
<td>$U_{\nu_i}$</td>
<td>$1.11\times10^{-5}$ m$^2$/s</td>
<td>1.57% $\nu_i$</td>
</tr>
</tbody>
</table>
Comparison with benchmark data

- Density $\rho$

$E = 4.9\%$ (reference data) and $E = 5.4\%$ (ErTco hydrometer)

Neglecting correlated bias errors:

$U_E \approx U_D = 1.30\%$

Data not validated:

$|E| \geq U_E$
Comparison with benchmark data

- Viscosity $\nu$

$E = 3.95\% \text{ (reference data)}$ and $E = 40.6\% \text{ (Cannon capillary viscometer)}$

Neglecting correlated bias errors:

\[ U_E \approx U_D = 1.57\% \text{(teflon)} \]
\[ U_E \approx U_D = 1.49\% \text{(steel)} \]

Data not validated (unaccounted bias error):

\[ |E| \geq U_E \]
References

References

- Proctor&Gamble, 1995, private communication.