NATURAL FREQUENCIES OF A BODY OF REVOLUTION VIBRATING TRANSVERSELY IN A FLUID

by
Louis Landweber

This research was sponsored by the Department of Acoustics and Vibration of the Naval Ship Research and Development Center under Naval Ship Systems Command Subproject SF 113 1106, Task 1360, Contract Nonr-3271(01) (X).

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Iowa Institute of Hydraulic Research
The University of Iowa
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1. Introduction

In a previous paper [1], a procedure for determining the natural frequencies of a body vibrating in a fluid was described and applied to a flexible circular cylinder. A more practical and more difficult application of the method, to the case of a body of revolution, is presented in the present work.

As was shown in [1], the natural frequencies are given by the eigenvalues of the potential energy matrix of the elastic body with respect to an inertia matrix, the latter being derived from the mass distribution of the body and the kinetic energy of the fluid. Thus two matrices must be obtained, and since the determination of the former is a problem in elasticity, and the determination of the latter one in hydrodynamics, these will be treated in separate sections. Then, in the final section, a particular body of revolution with prescribed elastic and inertial characteristics will be assumed, and its natural frequencies of vibration in air and in water will be calculated. For vibration in water, results obtained by means of strip theory, and by the present matrix technique will be compared.

2. Kinetic Energy of Fluid

Formulation of problem

Take the axis of symmetry of a body of revolution to extend from -1 to +1 along the x-axis of a cylindrical coordinate system \((x, r, \theta)\), and denote the equation of the body by

\[ r^2 = y^2 + z^2 = f(x) \]  \hspace{1cm} (1)

where

\[ y = r \cos \theta, \quad z = r \sin \theta \]  \hspace{1cm} (2)

Then, putting

\[ \frac{dr}{dx} = \tan \gamma \]  \hspace{1cm} (3)
we have for the direction cosines of the outward normal to the surface $S$ of the body

\[
\frac{\partial x}{\partial n} = -\frac{dr}{ds} = -\sin \gamma \\
\frac{\partial y}{\partial n} = \cos \theta \cos \gamma \\
\frac{\partial z}{\partial n} = \sin \theta \cos \gamma
\]

(4)

For a vertical vibration in the $y$-direction

\[
\dot{y} = V(x) e^{i \omega t}
\]

(5)

we may take the velocity potential (omitting the factor $e^{i \omega t}$) in the form

\[
\phi(x,r,\theta) = \phi(x,r) \cos \theta
\]

(6)

where $\phi(x,r,\theta)$ satisfies the Laplace equation and the boundary condition on the surface of the body

\[
\frac{\partial \phi}{\partial n} = V(x) \cos \theta \cos \gamma
\]

(7)

Then we have

\[
\frac{\partial \phi}{\partial n} = V(x) \cos \gamma
\]

(8)

An integral equation for the velocity potential

Consider Green's reciprocal formula

\[
\int_S \phi \frac{\partial \phi'}{\partial n} \, dS = \int_S \phi' \frac{\partial \phi}{\partial n} \, dS
\]

(9)

where $\phi$ and $\phi'$ are potential functions which vanish sufficiently rapidly at infinity. If we select $\phi'$ so that it is also of the form of (6),

\[
\phi'(x,r,\theta) = \phi'(x,r) \cos \theta
\]

then, by applying (8), we obtain from (9)

\[
\int_{-1}^1 r \phi \frac{\partial \phi'}{\partial n} \sec \gamma \, dx = \int_{-1}^1 r \phi' V(x) \, dx
\]

(10)
Let us take for \( \phi' \) the potential of a \( y \)-oriented doublet at the point \( x = \xi \) on the axis of symmetry,

\[
\phi' = \frac{V}{R^3} = \frac{R \cos \theta}{R^3}, \quad R = \left[ (x - \xi)^2 + r^2(x) \right]^{1/2}
\]

(11)

and hence

\[
\phi'(x,r) = \frac{r}{R^3}
\]

(12)

Then we have

\[
\frac{3\phi'}{3r} = \frac{2\phi'}{3x} + \frac{3\phi}{3r} \frac{3r}{3n} = - \frac{3\phi'}{3x} \sin \gamma + \frac{3\phi'}{3n} \cos \gamma
\]

\[
= \frac{3r(x - \xi)}{R^5} \sin \gamma + \frac{1}{R^3} - \frac{3r^2}{R^5} \cos \gamma
\]

(13)

and hence (10) becomes

\[
\int_{-1}^{1} r \frac{3(x - \xi)f'(x) + (x - \xi)^2 - 2r^2}{R^5} dx = \int_{-1}^{1} \frac{r^2}{R^3} V(x) dx
\]

(14)

where

\[
f'(x) = \frac{d}{dx} r^2(x)
\]

The kernel

\[
k(x,\xi) = \frac{1}{R^5} \left[ \frac{3}{2} f'(x)(x - \xi) + (x - \xi)^2 - 2r^2 \right]
\]

(15)

can be written in the form

\[
k(x,\xi) = \frac{1}{R^3} \left[ 3 \sin \alpha \cos \alpha \tan \gamma + 3 \cos^2 \alpha - 2 \right]
\]

(16)

where

\[
x - \xi = R \cos \alpha, \quad r(x) = R \sin \alpha
\]

(17)

Except near the ends of the body, \( \gamma \) is small and we have, approximately,

\[
k(x,\xi) \approx \frac{1}{R^3} (3 \cos^2 \alpha - 2) = \frac{\sin^3 \alpha}{r^3(x)} (3 \cos^2 \alpha - 2)
\]

(18)
For values of $x$ near $\xi$, $r(x)$ varies slowly (except near the ends of the body) as $a$ varies from $\pi/4$ to $3\pi/4$. In the course of this variation, $k(x, \xi)$ assumes approximately the same initial and final values, but decreases to a minimum at $a = \pi/2$:  

$$k(x, \xi) \approx \begin{cases} \frac{2}{\alpha^3}, & a = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \\ \frac{2}{r^3}, & a = \frac{\pi}{2} \end{cases}$$  

(19)

Thus, if $r(x) \ll 1$, the graph of $k(x, \xi)$ against $x$ would show a sharp trough in the neighborhood of $x = \xi$.

In order to examine the nature of the kernel near one end of the body, put

$$\xi = 1 - \xi', \quad x = 1 - x', \quad 0 < \xi', x' \ll 1$$

Also assume that in this neighborhood

$$r^2(x') = a_1 x' + a_2 x'^2 + \ldots$$

(20)

where $a_1$ and $a_2$ are of the same order of magnitude as $\xi'$. Then, noting that

$$r \tan \gamma = \frac{1}{2} f'(x)$$

and retaining terms to the second order, we can write (15) approximately in the form

$$k(x, \xi) \approx \frac{\frac{3}{2} (x' - \xi') a_1 + (x' - \xi')^2 - 2 a_1 x'}{[(x' - \xi')^2 + a_1 x']^{5/2}}$$

(21)

At $x' = 0, \xi'$, and $2 \xi'$ we obtain the values

$$k(x, \xi) \approx \begin{cases} \frac{\xi'' - \frac{3}{2} a_1}{\xi'^4}, & x' = 0 \\ \frac{2}{(a_1 \xi')^{3/2}}, & x' = \xi' \\ \frac{\xi' - \frac{5}{2} a_1}{\xi'^{3/2} (\xi' + 2 a_1)^{5/2}}, & x' = 2 \xi' \end{cases}$$

(21)
Each of these expressions is of the order of magnitude of $1/\xi'^3$.

The factor $r^2/R^3$ of the integrand in the right member of (14) can be analyzed in a similar way. We can write

$$\frac{r^2}{R^3} = \frac{\sin^2 \alpha}{r(x)}$$

Then, except near the ends of the body, we can write

$$\frac{r^2}{R^3} = \begin{cases} \frac{\sqrt{2}}{4r}, & \alpha = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \\ \frac{1}{r}, & \alpha = \frac{\pi}{2} \end{cases}$$

(22)

and when $x$ and $\xi$ are near 1,

$$\frac{r^2}{R^3} \approx \begin{cases} 0, & x' = 0 \\ \frac{1}{(a_1 \xi')^{1/2}}, & x' = \xi \\ \frac{2a_1}{\sqrt{\xi'} (\xi' + 2a_1)^{3/2}}, & x' = 2 \xi \end{cases}$$

(23)

Since $a_1$ is of the same order of magnitude as $\xi'$, each of the expressions in (23) is of the order of magnitude of $1/\xi'$.

Numerical treatment of integrals

If the body of revolution is elongated ($r_{\text{MAX}} << 1$), the foregoing analysis indicates that the integrands in (14) peak sharply in the neighborhood of $x = \xi$. Elsewhere the integrals in (14) can be readily replaced by a quadrature formula. This property suggests, however, that a good approximation to the value of the integral over a small interval about $x = \xi$ may be obtained as follows:

$$\int_{\xi_1}^{\xi_2} r(x)\phi(x)k(x,\xi)dx \approx \int_{\xi_1}^{\xi_2} \left\{ r(\xi)\phi(\xi) + (x - \xi) \frac{d}{d\xi} [r(\xi)\phi(\xi)] \right\} k'(x,\xi)dx$$

(24)
where \( k'(x, \xi) \) is an approximation to \( k(x, \xi) \) given by

\[
k'(x, \xi) = \frac{1}{R^3} - \frac{1}{R^3} \left[ 3f(\xi) + \frac{3}{2}(x - \xi)f'(\xi) \right]
\]

\[
R^2 = (x - \xi)^2 + f(\xi) + (x - \xi)f'(\xi) = (x - p)^2 + q
\]

\[
p = \xi - \frac{1}{2} f'(\xi), \quad q = f(\xi) - \frac{1}{4} f'(\xi)^2
\]

(25)

Evaluating the integrals occurring in (24), we obtain

\[
I_1 = \int_{\xi_1}^{\xi_2} k'(x, \xi)dx = \left[ -\frac{1}{q} \frac{x - p}{R'} - \frac{x - \xi}{R^3} \right]^{\xi_2}_{\xi_1}
\]

(26)

and

\[
I_2 = \int_{\xi_1}^{\xi_2} (x - \xi)(x, \xi)dx = \left[ \frac{(x - \xi)^2}{R^3} \right]^{\xi_2}_{\xi_1}
\]

(27)

in terms of which (24) may be written

\[
\int_{\xi_1}^{\xi_2} r(x)\phi(x)k(x, \xi)dx \approx r(\xi)\phi(\xi)I_1 + \frac{\partial}{\partial \xi} \left[ r(\xi)\phi(\xi)\right] I_2
\]

(28)

Similarly, the integral on the right of (14), evaluated in a small interval about \( \xi \), yields

\[
\int_{\xi_1}^{\xi_2} \frac{2}{R^3} V(x)dx \approx \int_{\xi_1}^{\xi_2} \frac{1}{R^3} \left\{ f(\xi)V(\xi) + (x - \xi) \frac{\partial}{\partial \xi} \left[ f(\xi)V(\xi) \right] \right\} dx
\]

\[
\approx f(\xi)V(\xi)J_1 + \frac{\partial}{\partial \xi} \left[ f(\xi)V(\xi) \right] J_2
\]

(29)

Here, by (25),

\[
J_1 = \int_{\xi_1}^{\xi_2} \frac{dx}{[(x - p)^2 + q]^{3/2}} = \frac{x - p}{qR'} \left| \frac{\xi_2}{\xi_1} \right|
\]

(30)

\[
J_2 = \int_{\xi_1}^{\xi_2} \frac{x - \xi}{[(x - p)^2 + q]^{3/2}} dx = -\frac{1}{R'} \left| \frac{\xi_2}{\xi_1} \right|
\]

(31)
Let us now replace the integrals in (14) by quadrature formulae. Subdivide the interval \(-1 \leq x \leq 1\) into \(n + 1\) segments of length \(\Delta x = 2/(n + 1)\) at the points \(x_i, i = 0, 1, 2, \ldots n + 1\), where \(x_0 = -1\) and \(x_{n+1} = 1\). Put \(\xi = x_j\) and \(\phi(x_i) = \phi_i, r(x_i) = r_i, k(x_i,x_j) = k_{ij}\), etc.

Using the trapezoidal rule, with \(\xi_1 = x_{j-1}\) and \(\xi_2 = x_{j+1}\) in (28), we obtain

\[
\int_{-1}^{1} r \phi k(x,\xi)dx \approx \Delta x \left[ \sum_{i=1}^{n} r_i \phi_i k_{i,j} \right] - \frac{1}{2} \left( r_{j-1} \phi_{j-1} k_{j-1,j} + r_{j} \phi_{j} k_{j,j} \right) + r_{j+1} \phi_{j+1} k_{j+1,j}
\]

where

\[
K_j = -\frac{1}{q_j} \left[ \frac{1}{R_{1j}^1} + \frac{1}{R_{2j}^1} - \frac{f_j'^2}{R_{1j}^1 R_{2j}^1 (R_{1j}^1 + R_{2j}^1)} \right] - \frac{1}{R_{1j}^3} - \frac{1}{R_{2j}^3}
\]

\[
P_j = \frac{1}{2} \left( \frac{1}{R_{1j}^3} - \frac{1}{R_{2j}^3} \right)
\]

\[
R_{1j}^2 = (\Delta x)^2 - f_j'^2 \Delta x + f_j', \quad R_{2j}^2 = (\Delta x)^2 + f_j'^2 \Delta x + f_j
\]

Similarly the right-hand member of (14) may be written

\[
\int_{-1}^{1} \frac{r^2}{R^3} V(x)dx \approx \Delta x \left[ \sum_{i=1}^{n} \frac{r_i}{R_{1j}^3} V_i \right] - \frac{1}{2} \left( \frac{f_j'^{-1}}{R_{j-1,j}^3} V_{j-1} + \frac{2f_j'}{R_{jj}^3} V_j + \frac{f_{j+1}}{R_{j+1,j}^3} V_{j+1} \right) + f_{j-1} V_{j-1} Q_j + f_j V_j S_j - f_{j+1} V_{j+1} Q_j
\]

where

\[
Q_j = -\frac{f_j}{R_{1j}^1 R_{2j}^1 (R_{1j}^1 + R_{2j}^1)} \Delta x
\]

\[
S_j = \frac{1}{q_j} \left[ \frac{1}{R_{1j}^1} + \frac{1}{R_{2j}^1} - \frac{f_j'^2}{R_{1j}^1 R_{2j}^1 (R_{1j}^1 + R_{2j}^1)} \right]
\]
Thus (14) may be expressed in the form

\[ \sum_{i=1}^{n} r_i \phi_i H_{ij} = \sum_{i=1}^{n} V_i W_{ij} \]  

(38)

where

\[ H_{ij} = \begin{cases} 
  k_{ij} & , \quad i \neq j - 1, j, j + 1 \\
  \frac{1}{2} k_{j-1,j} + p_j, & \quad i = j - 1 \\
  k_j, & \quad i = j \\
  \frac{1}{2} k_{j+1,j} - p_j, & \quad i = j + 1 
\end{cases} \]  

(39)

and

\[ W_{ij} = \begin{cases} 
  \frac{f_i}{R_{ij}^3}, & \quad i \neq j - 1, j, j + 1 \\
  \frac{1}{2} \frac{f_{i-1}}{R_{j-1,j}^3} + q_j, & \quad i = j - 1 \\
  s_j, & \quad i = j \\
  \frac{1}{2} \frac{f_{j+1}}{R_{j+1,j}^3} - q_j, & \quad i = j + 1 
\end{cases} \]  

(40)

**Added-mass matrix**

The maximum value of the kinetic energy of the fluid during a vibration cycle is given by

\[ 2T_F = -\int_S \phi \frac{\partial \phi}{\partial \eta} dS \]  

(41)

or, by (6) and (7),

\[ 2T_F = -\pi \rho \int_{-1}^{1} r \phi V(x) dx \]  

(42)
where \( \rho \) is the mass density of the fluid. By applying the trapezoidal rule, (42) can be expressed in the form

\[
2T_F = -\pi \rho \Delta x \sum_{j=1}^{n} r_j \phi_j V_j
\]  

(43)

We wish to eliminate the quantities \( r_j \phi_j \) in (43). For this purpose we multiply (38) by the inverse matrix \( H^{-1} \) of \( H \) such that

\[
\sum_{j=1}^{n} H^{-1}_{ij} H_{jk} = \delta_{ik}
\]

where \( \delta_{ik} \) is the Kronecker delta. This gives

\[
r_j \phi_j = \sum_{i} \sum_{k} W_{ik} H^{-1}_{kj} V_i
\]

(44)

which, substituted into (43), yields the quadratic form

\[
2T_F = \Delta x \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} V_i V_j
\]

\[
A_{ij} = -\frac{\pi \rho}{2} \sum_{k=1}^{n} \left[ W_{ik} H^{-1}_{kj} + W_{jk} H^{-1}_{ki} \right]
\]

(45)

where \( A_{ij} \) is the desired added-mass matrix.

Note that \( A_{ij} \) has the dimensions of mass per unit length. For a body of length \( L \), the corresponding added-mass matrix is given by

\[
A'_{ij} = \frac{L^2}{4} A_{ij}
\]

(45a)

It will be of interest later to compare the result in (45) with that from "strip theory," in which an added-mass coefficient of unity is assumed at each circular section,

\[
A_{ij}^{(s)} = \pi \rho f_i \delta_{ij}
\]

(45b)

3. Potential Energy of Elastic Body

Potential energy of bending

The potential energy of bending of a body of length \( L \) is [2]
\[ V_B = \int_0^L \frac{EI}{2} y''^2 dx \]  

(46)

where \( E \) is the modulus of elasticity, \( I \) is the moment of inertia of the section area about a diameter, and \( y'' = d^2 y / dx^2 \). The free-free end conditions are

\[ y''(0) = y''(L) = y''''(L) = 0 \]

(47)

Let us divide the interval \( 0 \leq x \leq L \) into \( n + 1 \) sections of length

\[ h = \frac{L}{n + 1} \]

(48)

at the points \( x_i \), \( i = 0, 1, 2, \ldots, n + 1 \), and put \( y_i = y(x_i) \), \( y''(x_i) = y_i'' \), \( EI(x_i) = G_i \). The free-free conditions will be satisfied approximately by putting

\[ y_0'' = y_1'' = y_n'' = y_{n+1}'' = 0 \]

(49)

Setting

\[ y_i'' = \frac{1}{h^2} (y_{i+1}'' - 2y_i'' + y_{i-1}'') \]

(50)

and applying the trapezoidal rule, (46) may be written as

\[ V_B = \frac{1}{2h^3} \sum_{i=2}^{n-1} (y_{i+1}'' - 2y_i'' + y_{i-1}'')^2 G_i \]

(51)

The \( n + 2 \) variables \( y_0, y_1, \ldots, y_n, y_{n+1} \) are not all independent since they are subject to the four conditions (49). We may use these conditions to express \( y_0, y_1, y_n, y_{n+1} \) in terms of \( y_2, y_3, \ldots, y_{n-1} \). Since \( y_0 \) and \( y_{n+1} \) do not appear in (51), we need consider only the elimination of \( y_1 \) and \( y_n \).

Applying the condition \( y_1'' = 0 \), we may write

\[ y \approx y_1 + a(x - x_1) + b(x - x_1)^3 \]

(52)

for the neighborhood of \( y_1 \). Then we have

\[ y_2 = y_1 + ah + bh^3 \]

\[ y_3 = y_1 + 2ah + 8bh^3 \]

\[ y_4 = y_1 + 3ah + 27bh^3 \]
Hence, eliminating \( a \) and \( b \), we obtain,

\[
y_1 = \begin{bmatrix} y_2 & 1 & 1 \\ y_3 & 2 & 8 \\ y_4 & 3 & 27 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 8 \\ 1 & 3 & 27 \end{bmatrix}
\]

or

\[
y_1 = \frac{1}{2} (5y_2 - 4y_3 + y_4)
\]  
\[(53)\]

Similarly

\[
y_n = \frac{1}{2} (5y_{n-1} - 4y_{n-2} + y_{n-3})
\]  
\[(54)\]

We now have, by (53) and (54),

\[
y_1 - 2y_2 + y_3 = \frac{1}{2} (y_2 - 2y_3 + y_4)
\]

\[
y_n - 2y_{n-1} + y_{n-2} = \frac{1}{2} (y_{n-1} - 2y_{n-2} + y_{n-3})
\]

and hence (51) becomes

\[
2V_B = \frac{1}{\hbar^3} \sum_{i=3}^{n-1} (y_{i+1} - 2y_i + y_{i-1})^2 G_i' = \frac{1}{\hbar^3} \sum_{i,j=2}^{n-1} h_{ij} y_i y_j
\]  
\[(55)\]

where

\[
G_3' = G_2 \quad G_{n-2}' = G_{n-2} + \frac{1}{4} G_{n-1}' \quad G_i' = G_i', \quad i = 4, 5, \ldots, n - 3
\]

and \( h_{ij} = h_{ji} \) and is nonzero only for

\[
h_{ii} = G_{i+1}' + 4G_i' + G_{i-1}', \quad i = 2, 3, \ldots, n - 1
\]

\[
h_{i,i+1} = -2(G_i' + G_{i+1}') \quad i = 2, 3, \ldots, n - 2
\]  
\[(56)\]

\[
h_{i,i+2} = G_{i+1}' \quad i = 2, 3, \ldots, n - 3
\]

In applying (56) we must set

\[
G_2' = G_3' = G_{n-1}' = G_n' = 0
\]  
\[(57)\]

since only \( G_3', G_4', \ldots, G_{n-2}' \) occur in (55).
An interesting property of the matrix \((h_{ij})\) is that it is singular, and that its rank does not exceed \(n - 4\).

In order to prove this property, consider the determinant \(|h_{ij}|, \ i,j = 2, 3, \ldots, n - 1\). We may replace the elements \(h_{2j}\) of the first row by the sum of the elements over all the rows. By (56) and (57) this yields

\[
\sum_{i=2}^{n-1} h_{ij} = G'_{j-1} - 2(G'_{j-1} + G'_j) + (G'_{j+1} + 4G'_j + G'_{j-1}) - 2(G'_j + G'_{j+1}) + G'_{j+1} = 0
\]

i.e. the elements of the first row are now all zeros. Next multiply the elements of the \(i\)-th row by \(i - 1\) and replace the elements of the last row by the sum of the resulting elements in all the rows, to obtain

\[
\sum_{i=2}^{n-1} (i - 2)h_{ij} = (j - 4)G'_{j-1} - 2(j - 3)(G'_{j-1} + G'_j) + (j - 2)(G'_{j+1} + 4G'_j + G'_{j-1}) - 2(j - 1)(G'_j + G'_{j+1}) + jG'_{j+1} = 0
\]

Thus the elements of the last row have also been replaced by zeros, showing that the rank of the matrix \((h_{ij})\) is at most \(n - 4\), as was stated.

We shall also need to eliminate \(V_1\) and \(V_n\) from the quadratic form for the kinetic energy

\[
2T = h \sum_{i,j=1}^{n} a_{ij} V_i V_j = h \sum_{i,j=2}^{n-1} b_{ij} V_i V_j
\]

by means of the relations corresponding to (53) and (54),

\[
\begin{align*}
V_1 &= \frac{1}{2} (5V_2 - 4V_3 + V_4) \\
V_n &= \frac{1}{2} (5V_{n-1} - 4V_{n-2} + V_{n-3})
\end{align*}
\]

We obtain:

\[
b_{ij} = a_{ij}, \quad 5 \leq i,j \leq n - 4
\]

\[
b_{22} = a_{22} + \frac{25}{4} a_{11} + 5a_{12} \quad b_{n-1,n-1} = a_{n-1,n-1} + \frac{25}{4} a_{nn} + 5a_{n,n-1}
\]

\[
b_{23} = a_{23} - 5a_{11} - 2a_{12} + \frac{5}{2} a_{13} \quad b_{n-1,n-2} = a_{n-1,n-2} - 5a_{nn} - 2a_{n,n-1} + \frac{5}{2} a_{n,n-2}
\]
\[ b_{24} = a_{24} + \frac{5}{4} a_{11} + \frac{1}{2} a_{12} + \frac{5}{2} a_{14} \quad b_{n-1,n-3} = a_{n-1,n-3} + \frac{5}{4} a_{nn} + \frac{1}{2} a_{n,n-1} + \frac{5}{2} a_{n,n-3} \]

\[ b_{33} = a_{33} + 4 a_{11} - 4 a_{13} \quad b_{n-2,n-2} = a_{n-2,n-2} + 4 a_{nn} - a_{n,n-2} \]

\[ b_{34} = a_{34} - a_{11} + \frac{1}{2} a_{13} - 2 a_{14} \quad b_{n-2,n-3} = a_{n-2,n-3} - a_{nn} + \frac{1}{2} a_{n,n-2} - 2 a_{n,n-3} \]

\[ b_{44} = a_{44} + \frac{1}{4} a_{11} + a_{14} \quad b_{n-3,n-3} = a_{n-3,n-3} + \frac{1}{4} a_{nn} + a_{n,n-3} \]

\[ b_{2j} = a_{2j} + \frac{5}{2} a_{ij} \quad b_{n-1,j} = a_{n-1,j} + \frac{5}{2} a_{nj} \]

\[ b_{3j} = a_{3j} - 2 a_{ij} \quad b_{n-2,j} = a_{n-2,j} - 2 a_{nj} \quad \quad \quad 5 \leq j \leq n-4 \]

\[ b_{4j} = a_{4j} + \frac{1}{2} a_{ij} \quad b_{n-3,j} = a_{n-3,j} + \frac{1}{2} a_{nj} \]

\[ b_{2,n-1} = a_{2,n-1} + \frac{5}{2} a_{1,n-1} + \frac{25}{4} a_{1n} + \frac{5}{2} a_{2n} \]

\[ b_{2,n-2} = a_{2,n-2} + \frac{5}{2} a_{1,n-2} - 5 a_{1n} - 2 a_{2n} \]

\[ b_{2,n-3} = a_{2,n-3} + \frac{5}{2} a_{1,n-3} + \frac{5}{4} a_{1n} + \frac{1}{2} a_{2n} \]

\[ b_{3,n-1} = a_{3,n-1} - 2 a_{1,n-1} - 5 a_{1n} + \frac{5}{2} a_{3n} \]

\[ b_{3,n-2} = a_{3,n-2} - 2 a_{1,n-2} + 4 a_{1n} - 2 a_{3n} \]

\[ b_{3,n-3} = a_{3,n-3} - 2 a_{1,n-3} - a_{1n} + \frac{1}{2} a_{3n} \]

\[ b_{4,n-1} = a_{4,n-1} + \frac{1}{2} a_{1,n-1} + \frac{5}{4} a_{1n} + \frac{5}{2} a_{4n} \]

\[ b_{4,n-2} = a_{4,n-2} + \frac{1}{2} a_{1,n-2} - a_{1n} - 2 a_{4n} \]

\[ b_{4,n-3} = a_{4,n-3} + \frac{1}{2} a_{1,n-3} + \frac{1}{4} a_{1n} + \frac{1}{2} a_{4n} \]

(60)
If $a_{ij}$ is a diagonal matrix, (60) becomes

$$\begin{aligned}
b_{22} &= a_{22} + \frac{25}{4} a_{11} \\
b_{n-1,n-1} &= a_{n-1,n-1} + \frac{25}{4} a_{nn} \\
b_{23} &= -5a_{11} \\
b_{n-1,n-2} &= -5a_{nn} \\
b_{24} &= \frac{5}{4} a_{11} \\
b_{n-1,n-3} &= \frac{5}{4} a_{nn} \\
b_{33} &= a_{33} + 4a_{11} \\
b_{n-2,n-2} &= a_{n-2,n-2} + 4a_{nn} \\
b_{34} &= -a_{11} \\
b_{n-2,n-3} &= -a_{nn} \\
b_{44} &= a_{44} + \frac{1}{4} a_{11} \\
b_{n-3,n-3} &= a_{n-3,n-3} + \frac{1}{4} a_{nn} \\
b_{ij} &= a_{ii} \delta_{ij}, \quad 5 \leq i, j \leq n - 4
\end{aligned}$$

(60a)

**Potential energy of bending and shear**

The shearing force $F$ is related to the bending moment $M$ by the relation

$$F = -\frac{dM}{dx}$$  \hspace{1cm} (61)

Let $y_B$ denote the deflection due to bending, $y_s$ that due to shear, so that the total deflection $y$ is given by

$$y = y_B + y_s$$  \hspace{1cm} (62)

Here $y_B$ satisfies the free-free conditions (49). Then we have

$$M = EIy''_B, \quad F = -EIy''_B$$  \hspace{1cm} (63)

The shearing force can also be expressed in the form

$$F = kA \sigma \beta, \quad \beta = y'_s$$  \hspace{1cm} (64)

where $A$ is the area of transverse section, $G$ is the shear modulus and $k$ is a numerical factor, given by $k = \frac{1}{2}$ for a circular section. Equating the expressions for $F$ in (63) and (64), and applying (62), we obtain the relation
\[ -EIy''_B + kAGy'_B = kAGy'_B \]  
\[ (65) \]

The potential energy of bending, given in (46), must now be replaced by

\[ V_B = \frac{1}{2} \int_0^L EIy''_B \, dx \]  
\[ (66) \]

The potential energy of shear is given by

\[ V_s = \int_0^L \int_0^L Fd\delta dx = \frac{1}{2} \int_0^L kAG\delta^2 dx \]

or, by (63) and (64),

\[ V_s = \frac{1}{2} \int_0^L \frac{(EI)^2}{kAG} y''_B^2 \, dx \]  
\[ (67) \]

Hence the total potential energy is

\[ V = V_B + V_s = \frac{1}{2} \int_0^L [EIy''_B^2 + \frac{(EI)^2}{kAG} y''_B^2] \, dx \]  
\[ (68) \]

The discretization procedure for \( V_B \) yields the same form as in (55)

\[ 2V_B = \frac{1}{h^3} \sum_{i=1}^{n-1} h_i \sum_{j=2}^{n} h_{ij} y_{Bi} y_{Bj} \]  
\[ (69) \]

For \( V_s \) we have by the trapezoidal rule

\[ 2V_s = h \sum_{i=1}^{n} H_i y''_{Bi} \quad H_i = \frac{G_i^2}{kGA_i} \]  
\[ (70) \]

Here \( i = 0 \) and \( n + 1 \) are excluded because of the free-free condition (47).

Since \( y''_B(0) = 0 \), we may assume, for the neighborhood of \( x = 0 \), that \( y_B''(x) \) is of the form

\[ y_B''(x) = cx^2 \]

and then

\[ y_B''(x) = 2cx \]
i.e. \( y_B^{mi}(x) \) is proportional to \( x \) near \( x = 0 \). Hence we have

\[
y_B^{mi} = \frac{1}{2} y_B^{m2}
\]

At \( i = 2.5 \) we may put

\[
y_{B2\frac{1}{2}}^{mi} = \frac{1}{h^3} (y_4 - 3y_3 + 3y_2 - y_1)_B
\]

We shall assume then

\[
y_B^{m1} = \frac{0.2}{h^3} (y_2 - 2y_3 + y_4)_B
\]

or, by (53),

\[
y_B^{m1} = \frac{0.2}{h^3} (y_2 - 2y_3 + y_4)_B
\]

and, by (71),

\[
y_B^{m2} = \frac{0.4}{h^3} (y_2 - 2y_3 + y_4)_B
\]

Similarly,

\[
y_B^{m1} = \frac{0.2}{h^3} (y_n - 2y_{n-1} + y_{n-2})_B
\]

\[
y_B^{m1} = \frac{0.4}{h^3} (y_n - 2y_{n-1} + y_{n-2})_B
\]

For \( y_B^{m1} \), \( i = 3, 4, \ldots, n - 2 \), we have the central-difference formula

\[
y_B^{m1} = \frac{1}{2h^3} (y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})_B
\]

Since \( y_{1B} \) and \( y_{nB} \) occur in these expressions for \( y_{B3}^{m1} \) and \( y_{B,n-2}^{m1} \), we again apply (53) and (54), which yield

\[
y_B^{m1} = \frac{1}{4h^3} (2y_5 - 5y_4 + 4y_3 - y_2)_B
\]

\[
y_B^{m1} = \frac{1}{4h^3} (2y_{n-4} - 5y_{n-3} + 4y_{n-2} - y_{n-1})_B
\]

Applying (76), we obtain

\[
h \sum_{i=4}^{n-3} y_{B1}^{m2} = \frac{1}{4h^5} \sum_{i,j=2}^{n-1} c_{ij} y_{Bi} y_{Bj}
\]
where $c_{ij} = c_{ji}^*$ and

$$
c_{ii} = H_{i-2} + 4H_{i-1} + 4H_{i+1} + H_{i+2}, \quad i = 2, 3, \ldots, n-1
$$

$$
c_{i,i+1} = -2(H_{i-1} + H_{i+2}), \quad i = 2, 3, \ldots, n-2
$$

$$
c_{i,i+2} = -H_{i+1}, \quad i = 2, 3, \ldots, n-3
$$

$$
c_{i,i+3} = 2(H_{i+1} + H_{i+2}), \quad i = 2, 3, \ldots, n-4
$$

$$
c_{i,i+4} = -H_{i+2}, \quad i = 2, 3, \ldots, n-5
$$

Since only $H_1, H_2, \ldots, H_{n-3}$ occur in (79), we must set in (80)

$$
H_0 = H_1 = H_2 = H_3 = H_{n-2} = H_{n-1} = H_n = H_{n+1} = 0
$$

To (79) we must now add

$$
h[H_1 y_{Bl}^2 + H_2 y_{B2}^2 + H_3 y_{B3}^2 + H_{n-2} y_{B,n-2}^2 + H_{n-1} y_{B,n-1}^2 + H_n y_{B,n}^2]
$$

From (72), (73), and (77) we have (with $H_1, H_2, H_3 \neq 0$)

$$
h(H_1 y_{Bl}^2 + H_2 y_{B2}^2 + H_3 y_{B3}^2)
$$

$$
= \frac{1}{h^5} \left[ (0.04H_1 + 0.16H_2)(y_2^2 + 4y_3^2 + y_4^2 - 4y_2 y_3 + 2y_2 y_4 - 4y_3 y_4) + \frac{H_3}{16} (4y_5^2 + 25y_4^2 + 16y_3^2 + y_2^2 - 8y_2 y_3 + 10y_2 y_4 - 4y_2 y_5 - 4y_3 y_4 + 16y_3 y_5 - 20y_4 y_5) \right]_B
$$

Hence, putting

$$
2V_s = \frac{1}{4h^5} \sum_{i,j=2}^{n-1} (c_{ij} + c_{ji}) y_{Bi} y_{Bj}
$$

we have, with $H_2' = 0.16 H_1 + 0.64 H_2$, 

(82)
\[c_{22} + d_{22} = \frac{H_2}{2} + \frac{H_3}{4} + H_4\]
\[c_{23} + d_{23} = -2H_2 - H_3 - 2H_4\]
\[c_{24} + d_{24} = H_2 + \frac{5}{4} H_3\]
\[c_{25} + d_{25} = -\frac{H_3}{2} + 2H_4\]
\[c_{33} + d_{33} = 4H_2 + 4H_3 + 4H_4 + H_5\]
\[c_{34} + d_{34} = -2H_2 - 5H_3 - 2H_5\]
\[c_{35} + d_{35} = 2H_3 - 4H_4\]
\[c_{44} + d_{44} = H_2 + \frac{25}{4} H_3 + 4H_5 + H_6\]
\[c_{45} + d_{45} = -\frac{5}{2} H_3 - 2H_6\]
\[c_{55} + d_{55} = H_3 + 4H_4 + 4H_6 + H_7\]

and similarly, with \(H'_{n-1} = 0.16H_n + 0.64H_{n-1}\),
\[c_{n-1,n-1} + d_{n-1,n-1} = \frac{H'_{n-1}}{n-1} + \frac{H_{n-2}}{4} + H_{n-3}\]
\[c_{n-1,n-2} + d_{n-1,n-2} = -2H'_{n-1} - H_{n-2} - 2H_{n-3}\]
\[c_{n-1,n-3} + d_{n-1,n-3} = \frac{H'_{n-1}}{n-1} + \frac{5}{4} H_{n-2}\]
\[c_{n-1,n-4} + d_{n-1,n-4} = -\frac{1}{2} H_{n-2} + 2H_{n-3}\]
\[c_{n-2,n-2} + d_{n-2,n-2} = 4H'_{n-1} + 4H_{n-2} + 4H_{n-3} + H_{n-4}\]
\[ \begin{align*}
    c_{n-2,n-3} + d_{n-2,n-3} &= -2H_{n-1} - 5H_{n-2} - 2H_{n-4} \\
    c_{n-2,n-4} + d_{n-2,n-4} &= 2H_{n-2} - 4H_{n-3} \\
    c_{n-3,n-3} + d_{n-3,n-3} &= H_{n-1} + \frac{25}{4} H_{n-2} + 4H_{n-4} + H_{n-5} \\
    c_{n-3,n-4} + d_{n-3,n-4} &= -\frac{5}{2} H_{n-2} - 2H_{n-5} \\
    c_{n-4,n-4} + d_{n-4,n-4} &= H_{n-2} + 4H_{n-3} + 4H_{n-5} + H_{n-6}
\end{align*} \] (84)

Furthermore
\[ d_{ij} = 0, \quad 5 < i \text{ or } j < n - 4 \] (85)
i.e. the elements of the potential energy matrix in (82), other than those given in (83) and (84), can be calculated from (80).

We shall now show that the matrix \((c_{ij} + d_{ij})\) has the same property as \((h_{ij})\); i.e., its rank does not exceed \(n - 4\). Applying (80), (83), (84) and (85), we see that the sum of the elements of the \(j\)-th column of \((c_{ij} + d_{ij})\) is zero. This is certainly the case for a column in which (72) is satisfied, since

\[
\sum_{i=2}^{n-1} c_{ij} = -H_{j-2} + 2(H_{j-2} + H_{j-1}) - 4H_{j-1} - 2(H_{j-2} + H_{j+1}) + H_{j-2} + 4H_{j-1} + 4H_{j+1} + H_{j+2} - H_{j+2} = 0
\]

For each column containing nonzero elements, \(d_{ij}\), one can verify that the sum of the elements also vanishes; e.g.

\[
\begin{align*}
    (c_{23} + d_{23}) + (c_{33} + d_{33}) + (c_{43} + d_{43}) + (c_{53} + d_{53}) + c_{63} + c_{73} \\
    = (-2H_2 - H_3 - 2H_4) + (4H_4' + 4H_3 + 4H_4 + H_5) + (-2H_2' - 5H_3 - 2H_5) \\
    + (2H_3 - 4H_4) + 2(H_4 + H_5) - H_5 = 0
\end{align*}
\]

Thus the elements of the first row of the determinant \(|c_{ij} + d_{ij}|\) may be replaced by zeros. Next, multiplying the elements of the \(i\)-th row by \(i - 1\),
and replacing the elements of the last row by the sum of the resulting elements in each column, we find that each sum is zero. This is readily verified for a column in which (72) is satisfied, since

\[
\sum_{i=2}^{n-1} (i - 2)c_{ij} = - (j - 6)H_{j-2} + 2(j - 5)(H_{j-2} + H_{j-1}) - 4(j - 4)H_{j-1}
\]

\[
- 2(j - 3)(H_{j-2} + H_{j+1}) + (j - 2)(H_{j-2} + 4H_{j-1} + 4H_{j+1} + 4H_{j+2}) - 2(j - 1)(H_{j-1} + H_{j+2})
\]

\[
- 4H_{j+1} + 2(j + 1)(H_{j+1} + H_{j+2}) - (j + 2)H_{j+2} = 0
\]

and also shown by inspection to be valid for columns containing nonzero elements \(d_{ij}\); e.g., for the case considered above,

\[
(4H_2 + 4H_3 + 4H_4 + H_5) + 2(-2H_2 - 5H_3 - 2H_4) + 3(2H_3 - 4H_4) + 8(H_4 + H_5) - 5H_5 = 0
\]

Thus the elements of the last row of the determinant \([c_{ij} + d_{ij}]\) may also be replaced by zeros. This completes the proof that the rank of the matrix \((c_{ij} + d_{ij})\) does not exceed \(n - 4\).

**Kinetic energy corresponding to bending and shear**

Expression (58) for the kinetic energy

\[
2T = \sum_{i,j=1}^{n} a_{ij}V_iV_j
\]

is a quadratic form in terms of the deflection velocities, and that for the potential energy in (82), is in terms of the deflections due to bending alone. In order to apply the present method for determining natural frequencies of vibration, we must express both the kinetic and potential energies in terms of the same variables; i.e., we must express either the \(y_{Bi}\) in (82) in terms of the \(y_i\), or the \(V_i\) in (58) in terms of \(V_{Bi}\). Let us select the latter alternative.

It will be convenient to write (65) in the approximate form

\[
y' \approx y' - \frac{EI}{kAG} y'' - \frac{d}{dx} \left( \frac{EI}{kAG} y_B \right)
\]

in which the last additional term is small near the body ends because of the free-free condition (47), and small over the remainder of the body, where \(EI/kAG\) is nearly constant. Integration of this equation now yields
\[ y = y_B - \frac{EI}{kGA} y_B'' \]  \quad (86)

in which the constant of integration is set equal to zero on the assumption that the end deflections of a free-free beam are unaffected by shear. In discrete form (86) becomes

\[ y_i = y_{Bi} - \lambda_i(y_{Bi,i-1} - 2y_{Bi} + y_{Bi,i+1}), \quad \lambda_i = \frac{EI_i}{kGA_i}, \quad i = 1, 2, \ldots n \]  \quad (87)

The free-free conditions (49) will be satisfied in (87) if we put

\[ \lambda_0 = \lambda_1 = \lambda_n = \lambda_{n+1} = 0 \]  \quad (88)

Differentiating (87) with respect to time now yields

\[ V_i = -\lambda_i V_{Bi,i-1} + (1 + 2\lambda_i) V_{Bi} - \lambda_i V_{Bi,i+1} \]  \quad (89)

Then we have

\[ V_i V_j = [\lambda_i \lambda_j V_{i-1} V_{j-1} + (1 + 2\lambda_i)(1 + 2\lambda_j) V_i V_j + \lambda_i \lambda_j V_{i+1} V_{j+1} - \lambda_i (1 + 2\lambda_i) V_{i-1} V_j - \lambda_j (1 + 2\lambda_j) V_{i+1} V_{j+1} + \lambda_i \lambda_j V_{i-1} V_{j+1} + \lambda_i \lambda_j V_{i+1} V_{j-1} ]_B \]

Hence, writing

\[ 2T = \sum_{i,j=1}^{n} a_{ij} V_i V_j = \sum_{i,j=1}^{n} a'_{ij} B_i B_j \]  \quad (90)

we obtain

\[ a'_{ij} = \lambda_{i+1} \lambda_j a_{i+1,j+1} + (1 + 2\lambda_i)(1 + 2\lambda_j) a_{ij} + \lambda_{i-1} \lambda_j a_{i-1,j-1} - \lambda_{i+1} (1 + 2\lambda_j) a_{i+1,j} - \lambda_j (1 + 2\lambda_j) a_{i,j+1} - \lambda_{i-1} (1 + 2\lambda_j) a_{i-1,j+1} \]

\[ - \lambda_j (1 + 2\lambda_j) a_{i,j-1} + \lambda_{i+1} \lambda_j a_{i+1,j-1} + \lambda_{i-1} \lambda_j a_{i-1,j-1} \]  \quad (91)

in which (88) is applicable.

Finally, after the \( a'_{ij} \) have been computed, the kinetic energy can be expressed in the form

\[ 2T = \sum_{i,j=1}^{n-1} b'_{ij} B_i B_j \]  \quad (92)
in which the \( b'_{ij} \) are given in terms of the \( a'_{ij} \) by means of formulae (60).

4. Characteristic Equations

For the total kinetic energy of the body \( T_B \) we can write

\[
2T_B = \int_0^L m(x)V^2 dx
\]

or in discrete form, by the trapezoidal rule,

\[
2T_b = h \sum_{i=1}^{n} m_i V_i^2
\]  
(93)

Hence, by (45a), we have for the total kinetic energy \( T \)

\[
2T = h \sum_{i,j=1}^{n} \left( m_i \delta_{ij} + A'_{ij} \right) V_i V_j
\]  
(94)

(a) Vibration in air, neglecting shear

The potential-energy matrix, given in (55), is \( \frac{1}{h^3} (h_{ij}) \). Since the kinetic-energy matrix for this case is \( h(m_i \delta_{ij}) \), application of the transformation (60a), with \( m_i \) substituted for \( a_{ii} \), yields the corresponding kinetic energy matrix, \( h b_{ij} \). Application of the matrix method described in [1] now gives the characteristic determinant equation

\[
|\omega^2 h b_{ij} - \frac{1}{h^3} h_{ij}| = 0
\]

\[
|p^2 b_{ij} - h_{ij}| = 0
\]  
(95)

where, with \( f \) the frequency of vibration,

\[
f = \frac{p}{2\pi h^2}, \quad p = h^2 \omega
\]  
(96)

By employing the procedure used to prove that the rank of \( (h_{ij}) \) does not exceed \( n - 4 \), the determinant in (95) is transformed into one with \( p^2 \) a factor of the elements of the first and last rows. Hence two of the eigenvalues \( p^2 \) of \( (h_{ij}) \) with respect to \( (b_{ij}) \) are zero, and there are at most \( n - 4 \) nonzero eigenvalues.
(b) Vibration in water (strip theory), neglecting shear

As in the previous case, the potential-energy matrix, \( \frac{1}{h^3} (h_{ij}) \), is given in (55). The added mass of a segment of radius \( r \) and thickness \( h \) is, according to strip theory, given by \( \pi \rho r^2 \), where \( \rho \) is the density of the fluid. Hence the kinetic-energy matrix is now

\[
(h(m_i + \pi \rho r_i^2) \delta_{ij})
\]

Substituting \( m_i + \pi \rho r_i^2 \) for \( a_{ii} \) in (60a) yields the corresponding matrix \( b_{ij} \), and formally the same characteristic equation (95). In this case also there are at most \( n - 4 \) nonzero eigenvalues.

(c) Vibration in water (added-mass matrix), neglecting shear

We now apply the kinetic-energy matrix given in (94). Substituting \( a_{ij} = m_i \delta_{ij} + A_i' \) into (60) yields the matrix \( b_{ij} \) and again, formally, the characteristic equation (95), which will have at most \( n - 4 \) nonzero eigenvalues.

(d) Vibration in air, with shear

By (69) and (82) we have for the potential-energy matrix

\[
\frac{1}{h^3} h_{ij} + \frac{1}{4h^5} (c_{ij} + d_{ij})
\]

From the kinetic-energy matrix \( (a_{ij}) = m_i \delta_{ij} \) we first obtain \( (a'_{ij}) \) from (91), and then transform the \( (a'_{ij}) \) to \( (b'_{ij}) \) by (60). The resulting characteristic equation,

\[
|p^2 b'_{ij} - (h_{ij} + \frac{c_{ij} + d_{ij}}{4h^2})| = 0
\]

again has at most \( n - 4 \) eigenvalues, as can be shown by the same procedure as was applied to (95).

(e) Vibration in water (strip theory), with shear

The potential-energy matrix is identical to that in the previous case, and the kinetic-energy matrix is the same as in (b). By successive transformations of \( a_{ij} = (m_i + \pi \rho r_i^2) \delta_{ij} \) through (91) and (60) we obtain
(b') which is then substituted into the characteristic equation (97). Again this has at most \( n - 4 \) nonzero eigenvalues.

(f) Vibration in water (added-mass matrix), with shear

The potential-energy matrix is the same as in cases (d) and (e), and the kinetic-energy matrix is \( (a_{ij}') = (m_i \delta_{ij} + A_{ij}') \). Transforming this into \( (b_{ij}') \) by (91) and (60) gives the final characteristic equation (96), which also has at most \( n - 4 \) nonzero eigenvalues.

5. Computer Programs

Computer programs for calculating the natural frequencies of a given body are given in Appendix 1. All programs are in double-precision arithmetic.

Program 1 yields the added-mass matrix \( a_{ij} \) of equation (145a) in mass units based on the long ton (2240 lb) as the unit of weight. Note that \( i,j \) vary from 1 to 19 in Program 1, corresponding to the division of the body into 20 sections. The constant FO between items 11 and 12 is the quantity

\[
FO = \frac{1.94 L^3}{2240 \cdot 8}
\]

where \( L \) is the length of the body, taken to be 66.37 ft.

Program 2 yields the mass matrices \( b_{ij} \) of (95) for the three cases (a) vibration in air, (b) vibration in water, strip theory, (c) vibration in water, added-mass matrix, all with shear neglected. The corresponding mass matrices \( b_{ij}' \) when the effects of shear are included are given in Program 3.

Program 4 gives both the potential-energy matrix \( h_{ij} \) of (55) and (95) when shear is neglected and that for shear, \( (c_{ij} + d_{ij})/4h^2 \), of (97).

Finally the natural frequencies for the 6 cases are obtained from Program 5, in which the eigenvalues of the characteristic equations (95) and (97) are derived by calling for an available subroutine in IBM Manual "System/360 Scientific Subroutine Package, Number H20-0205-3."

6. Numerical Example

The foregoing procedures were applied to an assumed body.
The added-mass matrix \( a_{ij} \) of (45a) is given in Table 1. Kinetic-energy matrices \( b_{ij} \) of (58) and (95) for the three cases of vibration in air, in water by strip theory, and in water, employing the added-mass matrix, all with shear neglected, are given in Tables 2a, 2b and 2c, respectively. The corresponding kinetic-energy matrices \( b'_{ij} \) of (92) and (97), with shear taken into account, are given in Tables 3a, 3b and 3c. Potential-energy matrices for bending alone, \( h_{ij} \) of (55) and (95), and for shear alone, \((c_{ij} + d_{ij})/4h^2\) of (82) and (97), are given in Tables 4a and 4b.

The resulting values of the natural frequencies for the aforementioned six conditions are given in Table 5. The effects of shear are seen to be important, especially for the higher modes. Frequencies obtained by means of the added-mass matrix are somewhat lower than those obtained by means of strip theory, the percentage difference decreasing as the order of the mode increases, contrary to the effect of shear.

An estimate of the errors in the values of the natural frequencies due to the finite order of the matrix may be derived by applying Fig. 2 of Ref. [1]. The magnitudes of the errors, shown in Table 6, indicate that the results for the first six modes should be reasonably accurate.
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**KINETIC ENERGY MATRIX FOR BENDING -VIBRATIONS IN AIR -**

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Table 2b

KINETIC ENERGY MATRIX FOR BENDING -VIBRATIONS IN WATER- STRIP THEORY

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Table 3a

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Table 3b

KINETIC ENERGY MATRIX FOR BENDING AND SHEAR -VIBRATIONS IN WATER-STRIP THEORY

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-7461250. & 32927600. & -36010300. & -32927600. & 88381800. & 86947300. \\
0. & -9002570. & 36010300. & -36010300. & -35805100. & -8899980. \\
0. & 0. & -9002570. & 36010300. & 35805100. & 30468900. \\
0. & 0. & 0. & -8899980. & 30468900. & 6334470. \\
0. & 0. & 0. & 0. & 0. & 0. \\
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### TABLE 5

**NATURAL FREQUENCIES, C.P.S.**

| MODE | Bending | | Bending and Shear | |
|------|---------|-----------------|-------------------|
|      | In Air  | Strip Theory   | Added-Mass Matrix| In Air  | Strip Theory | Added-Mass Matrix |
| 1    | 20.8    | 14.0            | 12.3              | 18.3    | 12.3          | 10.9              |
| 2    | 51.9    | 34.1            | 30.5              | 41.4    | 27.8          | 25.0              |
| 3    | 86.7    | 58.4            | 53.9              | 61.4    | 41.8          | 38.5              |
| 4    | 132.5   | 87.7            | 81.7              | 80.7    | 54.6          | 51.4              |
| 5    | 195.8   | 128.2           | 120.1             | 88.6    | 59.5          | 55.9              |
| 6    | 268.0   | 174.0           | 164.4             | 91.0    | 61.4          | 57.8              |
| 7    | 344.1   | 225.1           | 214.6             | 102.6   | 70.1          | 66.7              |
| 8    | 431.0   | 282.8           | 271.4             | 107.9   | 73.7          | 69.8              |
| 9    | 523.4   | 343.5           | 331.0             | 122.8   | 80.9          | 76.9              |
| 10   | 616.4   | 407.0           | 392.5             | 127.8   | 84.9          | 81.0              |
| 11   | 704.5   | 466.1           | 447.8             | 139.7   | 88.9          | 85.2              |
| 12   | 782.9   | 521.5           | 499.2             | 144.8   | 95.0          | 91.3              |
| 13   | 884.1   | 592.7           | 564.8             | 152.7   | 96.3          | 93.2              |
| 14   | 938.8   | 615.9           | 582.3             | 166.5   | 107.8         | 103.6             |
| 15   | 1044.8  | 679.8           | 640.4             | 176.3   | 109.9         | 106.2              |
TABLE 6

ERRORS IN NATURAL FREQUENCIES

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<td>5.0</td>
<td>6.4</td>
<td>8.1</td>
<td>10.0</td>
<td>12</td>
<td>14</td>
<td>17</td>
<td>19</td>
<td>22</td>
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7. Acknowledgements

The computer programs given in the Appendix were prepared by Matilde Macagno, to whom the author here expresses his appreciation for successfully completing so complex a task. Discussions with Professor Kwan Rim of the University of Iowa Mechanics and Hydraulics Department were useful in the formulation of the expressions for potential energy.

8. References


PROGRAM FOR ADDEC MASS-MATRIX

REAL*8 X(20), F(20), FP(20), BE(19), C(19, 19), W(19, 19), CA(19, 19), DX
1(2C), EL(19), CK(19), MF(19), LE(19), EM(19), CK2(19), D(19, 19)

H = 3.400
DX(1) = 0.77896D0
D0 2 I = 1, 19
2 DX(I+1) = 0.102456D0
DX(20) = 0.77896D0
100 REAC(5, 1) X,F,FP
1 FORMAT(8F10.8)
WRITE(6,4)(X(I), F(I), FP(I), I=1, 19)
4 FORMAT(45H X(I) F(I) FP(I) /3F15.8)
DC 10 J = 1, 19
G =dx(J+1) + dx(J)
FF = FP(J) * FP(J)
BE(J) = F(J) - FF
D1 = SQRT(dx(J+1)*dx(J+1)+2.0D0* FP(J)) + F(J)
D2 = SQRT(dx(J)*dx(J)-2.0D0* FP(J)) + F(J)
D13 = D1**3
D23 = D2*3
F1 = (dx(J+1)* FP(J))/C1 +(dx(J) - FP(J)) / D2) / BE(J)
F2 = (dx(J+1) /D13) + (dx(J) / D23)
F3 = F1* (F(J) - 2.0D0 * FF)
F4 = 2.0D0 * FP(J) * ((1.0D0/D1) - (1.0D0/D2))
EL(J) = F3 - F4
CK(J) = F1 + F2
H1 = DX(J+1) / (DX(J) * G)
H1 = DX(J) / (DX(J+1) * G)
H3 = (PI* J+1) / (DX(J+1) * DX(J))
CK2(J) = -(DX(J+1)*DX(J+1)/D13 - DX(J)*DX(J)/D23)
U1 = CX(J+1) + FP(J) + D1
U2 = D2 + FP(J) - DX(J)
E3 = ALOG(U1/U2)
E21 = 2.0D0 * F(J) * (F(J) - 2.0D0 * FF)
E22 = FP(J)* (3.0D0* F(J) - 4.0D0* FF)
E2 = (E21 + E22 * DX(J+1))/(2.0D0*BE(J) * D1)
E4 = (E21 - E22 * DX(J))/ (2.0D0*BE(J) * D2)
EM(J) = FP(J) * E3 - E2 + E4
DO 9 I = 1, 19
T1 = X(I) - X(J)
T2 = T1 * T1
DN = T2 + F(I)
W1(J) = 0.5D0*F(I) / (DN**1.5)
C(J,J) = 0.5D0*(3.0D0*T1*FP(I)+T2-2.0D0*F(I))/(DN**2.5)
K = J - 1
APPENDIX

L = J + 1
IF(I.EQ.K) GO TO 7
IF(I.EQ.L) GO TO 8
U = DX(I) + DX(I+1)
W(I,J) = W(I,J) * U
C(I,J) = C(I,J) * U
GO TO 9
7 W(K,J) = W(K,J) * DX(K) - E(EM,J) * H1
C(K,J) = C(K,J) * DX(K) - CK2(J) * H1
GO TO 9
C(L,J) = C(L,J) * DX(L+1) + CK2(J) * H2
9 CONTINUE
W(J,J) = EL(J) + EM(J) * H3
10 C(J,J) = -CK(J) + CK2(J) * H3
CALL MINV(C,19,DE,LE,ME)
DO 11 I = 1,19
DO 11 J = 1,19
CA(I,J) = 0.0
DO 11 K = 1,19
11 CA(I,J) = (CA(I,J) + W(I,K) * C(K,J) * (DX(J) + DX(J+1)))
PI = 1.570810700
FO = 31.650400 / H
DO 12 I = 1,19
DO 12 J = 1,19
C(I,J) = -(CA(I,J) + CA(J,I)) * PI * FO
12 C(J,J) = D(I,J)
WRITE(6,13) (D(I,J),J=1,10),I=1,19
13 FORMAT(27H ADDED MASS MATRIX A(I,J) /(10F12.6))
WRITE(6,14) (C(I,J),J=11,19),I=1,19
14 FORMAT(1H 9F12.6)
16.00 CONTINUE
WRITE(7,20) (D(I,J),J=1,7),I=1,19
20 FORMAT(7D11.6)
WRITE(7,20) (C(I,J),J=8,14),I=1,19
WRITE(7,21) (D(I,J),J=15,19),I=1,19
21 FORMAT(5D11.6)
CALL EXIT
END
PROGRAM 2

COMPUTATION OF MATRICES OF KINETIC ENERGY FOR BENDING A(I,J) AND B(I,J)

REAL*8 EM(19), AM(19), D(19,19), E(19,19), F(19,19)
II = 1
READ(5,1) EM, AM
1 FORMAT(7D11.6)
READ(5,1) ((D(I,J), J = 1, 7), I = 1, 19)
READ(5,1) ((U(I,J), J = 8,14), I = 1, 19)
READ(5,1) ((C(I,J), J =15,19), I = 1, 19)
7 FORMAT(5D11.6)
DO 3 I = 1, 19
DO 2 J = 1, 19
2 E(I,J) = 0.000
3 E(I,I) = EM(I)
100 DO 11 I = 1, 19
   DO 11 J = 1, 19
11 F(I,J) = E(I,J)
   DO 12 J = 5, 15
      F(2,J) = E(2,J) + 2.500 * E(1,J)
      F(3,J) = E(3,J) - 2.000 * E(1,J)
      F(4,J) = E(4,J) + 0.500 * E(1,J)
      F(J,18) = E(18,J) + 2.500 * E(19,J)
      F(J,17) = E(17,J) - 2.000 * E(19,J)
12 F(J,16) = E(16,J) + 0.500 * E(19,J)
      F(2,2) = E(2,2) + 6.2500 * E(1,1) + 5.000 * E(1,2)
      F(2,3) = E(2,3) - 5.000 * E(1,1) - 2.000*E(1,2) + 2.500* E(1,3)
      F(2,4) = E(2,4) + 1.2500*E(1,1) + 0.500* E(1,2) + 2.500*E(1,4)
      F(3,3) = E(3,3) + 4.000*E(1,1) - 4.000* E(1,3)
      F(3,4) = E(3,4) - E(1,1) + 0.500*E(1,3) - 2.000*E(1,4)
      F(4,4) = E(4,4) + 0.2500*E(1,1) + E(1,4)
      F(16,16) = E(16,16) + 0.2500*E(19,19) + E(19,16)
      F(16,17) = E(17,16) - E(19,19) + 0.500*E(19,17) - 2.000*E(19,16)
      F(17,17) = E(17,17) + 4.000*E(19,19) - E(19,17)
      F(16,18) = E(18,16) + 1.2500*E(19,19) + 0.500*E(19,18) + 2.500*E(19,16)
      F(17,18) = E(18,17) - 5.000*E(19,19) - 2.000*E(19,18) + 2.500*E(19,17)
      F(18,18) = E(18,18) + 6.2500*E(19,19) + 5.000*E(19,18)
      F(2,18) = E(2,18) + 2.500*E(1,18) + 6.2500*E(1,19) + 2.500*E(2,19)
      F(2,17) = E(2,17) + 2.500*E(1,17) - 5.000*E(1,19) - 2.000*E(2,19)
      F(2,16) = E(2,16) + 2.500*E(1,16) + 1.2500*E(1,19) + 0.500*E(2,19)
      F(3,18) = E(3,18) - 2.000*E(1,18) - 5.000*E(1,19) + 2.500*E(3,19)
      F(3,17) = E(3,17) - 2.000*E(1,17) + 4.000*E(1,19) - 2.000*F(3,19)
      F(3,16) = E(3,16) - 2.000*E(1,16) - E(1,19) + 0.500*E(2,19)
      F(4,18) = E(4,18) + 0.500*E(1,18) + 1.2500*E(1,19) + 2.500*E(4,19)
      F(4,17) = E(4,17) + 0.500*E(1,17) - E(1,19) - 2.000*E(4,19)
      F(4,16) = E(4,16) + 0.500*E(1,16) + 0.2500*E(1,19) + 0.500*E(4,19)
   DO 13 I = 1, 18
J = I + 1
DO 13 K = J, 19
13 F(K, I) = F(I, K)
DO 14 I = 2, 18
    I1 = I - 1
DO 14 J = 2, 18
    J1 = J - 1
14 F(I1, J1) = F(I, J)
    WRITE(6, 5)((F(I, J), J = 1, 9), I = 1, 17)
    FORMAT(1H 9D12.5)
    WRITE(6, 8)((F(I, J), J = 10, 17), I = 1, 17)
    FORMAT(1H 8D12.5)
    WRITE(7, 21)((F(I, J), J = 1, 5), I = 1, 17)
21 FORMAT(5D15.6)
    WRITE(7, 21)((F(I, J), J = 6, 10), I = 1, 17)
    WRITE(7, 21)((F(I, J), J = 11, 15), I = 1, 17)
    WRITE(7, 22)((F(I, J), J = 16, 17), I = 1, 17)
22 FORMAT(2D15.6)
    II = II + 1
    IF(II .EQ. 3) GO TO 26
    IF(II .GT. 3) GO TO 30
    DO 25 I = 1, 19
    DO 24 J = 1, 19
25 E(I, J) = 0.000
24 E(I, I) = EM(I) + AM(I)
    GO TO 100
26 DO 28 I = 1, 19
    DO 27 J = 1, 19
27 E(I, J) = D(I, J)
28 E(I, I) = C(I, I) + EM(I)
    GO TO 100
30 CONTINUE
    CALL EXIT
END
PROGRAM 3

PROGRAM OF KINETIC ENERGY FOR BENDING AND SHEAR A*(I,J) AND B*(I,J)

REAL*8 G(19),H(19),AL(19),BE(19),EM(19),AM(19),B(19,19),C(19,19),
LE(15,19),F(19,19)
II = 1
REAC(5,1) EM,AM
1 FORMAT (7D11.6)
REAC(5,2) G,H
2 FORMAT(7D10.0)
READ(5,1)((C(I,J),J= 1, 7),I=1,19)
READ(5,1)((D(I,J),J= 8,14),I=1,19)
READ(5,7)((C(I,J),J=15,19),I=1,19)
7 FORMAT(5C11.6)
DO 102 I = 1,19
BE(I) = 0.000
AL(I) = C.000
DO 101 J = 1,19
101 B(I,J) = 0.000
1C2 B(I,I) = EM(I)
DX2 = 11.5600
DO 105 I = 2,18
AL(I) = - H(I) / (G(I) * DX2)
1C5 BE(I) = 1.000 - 2.000 * AL(I)
BE(I) = 1.000
BE(19) = 1.000
120 DO 1C8 I = 2,18
IP = I + 1
IM = I - 1
E(I,1) = AL(2) * AL(IP) * B(2, IP) + BE(I) * B(1, I) + AL(2) * BE(I) * B(2, I) + AL(1) * B(IP) * B(1, IP) + AL(IM) * B(I, IM) + AL(2) * AL(IM) * B(2, IM)
E(I,1) = E(1,1)
E(I,19) = BE(I) * B(I, 19) + AL(IM) * AL(18) * B(IM, 18) + AL(IP) * B(IP, 19) + AL(IM) * B(IP, 19)
E(19,1) = E(I,19)
DO 1C8 J = 2,18
JP = J + 1
JM = J - 1
1C8 F(I,J) = E(I,J)
E(I,1) = AL(2) * AL(2) * B(2, 2) + B(1, 1) + 2.000 * AL(2) * B(1, 2)
E(19,19) = AL(18) * AL(18) * B(18, 18) + E(19,19) + 2.000 * AL(18) * B(18, 19)
E(1,19) = B(1, 19) + AL(2) * B(2, 19) + AL(18) * B(1, 18) + AL(2) * AL(18) * B(2, 18)
DO 12 J = 5,15

F(2, J) = E(2, J) + 2.500 * E(1, J)
F(3, J) = E(3, J) - 2.000 * E(1, J)
F(4, J) = E(4, J) + C.500 * E(1, J)
F(J, 18) = E(18, J) + 2.500 * E(19, J)
F(J, 17) = E(17, J) - 2.000 * E(19, J)

12 F(J, 16) = E(16, J) + C.500 * E(19, J)
F(2, 1) = E(2, 1) + 6.2500 * E(1, 1) + 5.000 * E(1, 2)
F(2, 3) = E(2, 3) - 5.000 * E(1, 1) - 2.000*E(1, 2) + 2.500* E(1, 3)
F(2, 4) = E(2, 4) + 1.2500*E(1, 1) + C.500*E(1, 2) + 2.500*E(1, 4)
F(3, 3) = E(3, 3) + 4.0000*E(1, 1) - 4.0000*E(1, 3)
F(3, 4) = E(3, 4) - E(1, 1) + 0.5000*E(1, 3) - 2.0000*E(1, 4)
F(4, 4) = E(4, 4) + 0.2500*E(1, 1) + E(1, 4)
F(16, 16) = E(16, 16) + 0.2500*E(19, 19) + E(19, 16)
F(16, 17) = E(17, 16) - E(19, 19) + 0.5000*E(19, 17) - 2.0000*E(19, 16)
F(17, 17) = E(17, 17) + 4.0000*E(19, 19) - E(19, 17)
F(16, 18) = E(18, 16) + 1.2500*E(19, 19) + 0.5000*E(19, 18) + 2.5000*E(19, 16)
F(17, 18) = E(18, 17) - 5.0000*E(19, 19) - 2.0000*E(19, 18) + 2.5000*E(19, 17)
F(18, 18) = E(18, 18) + 6.2500*E(19, 19) + 5.0000*E(19, 18)
F(2, 18) = E(2, 18) + 2.5000*E(1, 18) + 6.2500*E(1, 19) + 2.5000*E(2, 19)
F(2, 17) = E(2, 17) + 2.5000*E(1, 17) - 5.0000*E(1, 19) - 2.0000*E(2, 19)
F(2, 16) = E(2, 16) + 2.5000*E(1, 16) + 1.2500*E(1, 19) + 0.5000*E(2, 19)
F(3, 18) = E(3, 18) - 2.0000*E(1, 18) + 5.0000*E(1, 19) + 2.5000*E(3, 19)
F(3, 17) = E(3, 17) - 2.0000*E(1, 17) + 4.0000*E(1, 19) - 2.0000*E(3, 19)
F(3, 16) = E(3, 16) - 2.0000*E(1, 16) + E(1, 19) + 0.5000*E(2, 19)
F(4, 18) = E(4, 18) + 0.5000*E(1, 18) + 1.2500*E(1, 19) + 2.5000*E(4, 19)
F(4, 17) = E(4, 17) + 0.5000*E(1, 17) - E(1, 19) - 2.0000*E(4, 19)
F(4, 16) = E(4, 16) + 0.5000*E(1, 16) + 0.2500*E(1, 19) + 0.5000*E(4, 19)

DO 13 I = 1, 18
  J = I + 1
DO 13 K = J, 19
13 F(K, I) = F(I, K)
DO 14 I = 2, 18
  I1 = I - 1
DO 14 J = 2, 18
  J1 = J - 1
14 F(I1, J1) = F(I, J)
  WRITE(6, 5)((F(I, J), J = 1, 9), I = 1, 17)
5 FORMAT(1H 9D12.5)
  WRITE(6, 8)((F(I, J), J = 10, 17), I = 1, 17)
8 FORMAT(1H 8D12.5)
  WRITE(7, 21)((F(I, J), J = 1, 5), I = 1, 17)
21 FORMAT(5D15.6)
  WRITE(7, 21)((F(I, J), J = 6, 10), I = 1, 17)
  WRITE(7, 21)((F(I, J), J = 11, 15), I = 1, 17)
  WRITE(7, 22)((F(I, J), J = 16, 17), I = 1, 17)
22 FORMAT(20D15.6)
  II = II + 1
  IF(II.EQ.3) GO TO 26
  IF(II.GT.3) GO TO 30
APPENDIX

DO 25 I = 1,19
DO 24 J = 1,19
24 B(I,J) = 0.000
25 B(I,I) = EM(I) + AM(I)
   GO TO 120
26 DO 28 I = 1,19
27 B(I,J) = C(I,J)
28 P(I,I) = C(I,I) + EM(I)
   GC TO 120
30 CONTINUE
   CALL EXIT
   END
PROGRAM 4

COMPUTATION OF MATRICES OF POTENTIAL ENERGY FOR BENDING AND SHEAR

C A(I,J) = MATRIX OF POTENTIAL ENERGY FOR BENDING
C C(I,J) = MATRIX OF POTENTIAL ENERGY FOR SHEAR
REAL*8 H(19), G(19), C(19,19), A(19,19), U
II = 1
READ(5,1) G, H
1 FORMAT(7D10.0)
DO 2 J = 1,19
DO 2 I = 1,19
C(I,J) = 0.0DDO
2 A(I,J) = 0.0DDO
DO 3 I = 2,18
M = I + 1
N = I - 1
A(I,I) = (G(M) + 4.0DDO*G(I) + C(N))
U = - 2.0DDO*(G(I) + G(M))
A(I,M) = U
3 A(M,M) = G(I)
I = 2
DO 102 J = 6,18
I1 = I + 1
I2 = I + 2
I3 = I + 3
I4 = I + 4
C(I1,J) = - H(I2)
C(I1,J) = 2.0DDO * (H(I2) + H(I3))
C(I2,J) = - 4.0DDO * H(I3)
C(I3,J) = - 2.0DDO * (H(I2) + H(I4+1))
C(I4,J) = H(I2) + 4.0DDO * (H(I3) + H(I4 + 1)) + H(I4 + 2)
I = I + 1
102 CONTINUE
H(2) = 46717220.0DDO
H(18) = 22511850.0DDO
C(2,2) = H(2) + 0.25DDO*H(3) + H(4)
C(2,3) = - 2.0DDO*H(2) - H(3) - 2.0DDO*H(4)
C(2,4) = H(2) + 1.25DDO * H(3)
C(2,5) = - 0.5DDO * H(3) + 2.0DDO * H(4)
C(3,3) = 4.0DDO * (H(2) + H(3) + H(4)) + H(5)
C(3,4) = - ( 2.0DDO * (H(2) + H(5)) + 5.0 * H(3))
C(3,5) = 2.0DDO * H(3) - 4.0DDO * H(4)
C(4,4) = H(2) + 6.25DDO * H(3) + 4.0DDO * H(5) + H(6)
C(4,5) = - ( 2.5DDO * H(3) + 2.0DDO * H(6))
C(5,5) = H(3) + 4.0DDO * (H(4) + H(6)) + H(7)
C(18,18) = H(18) + C.25DDO * H(17) + H(16)
C(17,18) = - ( 2.0DDO * (H(18) + H(16)) + H(17))
APPENDIX

\[ C(16,18) = f(18) + 1.25D0 \times H(17) \]
\[ C(15,18) = -0.5D0 \times H(17) + 2.0D0 \times H(16) \]
\[ C(17,17) = 4.0D0 \times (H(18) + H(17) + H(16)) + H(15) \]
\[ C(16,17) = -(2.0D0 \times (H(18) + H(15)) + 5.0D0 \times H(17)) \]
\[ C(15,17) = 2.0D0 \times H(17) - 4.0D0 \times H(16) \]
\[ C(16,16) = H(18) + 6.25D0 \times H(17) + 4.0D0 \times H(15) + H(14) \]
\[ C(15,16) = -(2.5D0 \times H(17) + 2.0D0 \times H(14)) \]
\[ C(15,15) = H(17) + 4.0D0 \times (H(16) + H(14)) + H(13) \]

DC 103 I = 1, 18
J = I + 1
DO 103 K = J, 19
A(K,I) = A(I,K)
103 C(K,I) = C(I,K)
DO 104 I = 2, 18
II = I - 1
DO 104 J = 2, 18
J1 = J - 1
A(I1,J1) = A(I,J)
104 C(I1,J1) = C(I,J) / 46.2400
WRITE(6,5)((A(I,J), J = 1, 9), I = 1, 17)
5 FORMAT(1H 9D12.5)
WRITE(6,8)((A(I,J), J = 10, 17), I = 1, 17)
8 FORMAT(1H 8D12.5)
WRITE(6,5)((C(I,J), J = 1, 9), I = 1, 17)
WRITE(6,8)((C(I,J), J = 10, 17), I = 1, 17)
WRITE(7,21)((A(I,J), J = 1, 5), I = 1, 17)
21 FORMAT(5D15.6)
WRITE(7,21)((A(I,J), J = 6, 10), I = 1, 17)
WRITE(7,21)((A(I,J), J = 11, 15), I = 1, 17)
WRITE(7,22)((A(I,J), J = 16, 17), I = 1, 17)
22 FORMAT(20D15.6)
WRITE(7,21)((C(I,J), J = 1, 5), I = 1, 17)
WRITE(7,21)((C(I,J), J = 6, 10), I = 1, 17)
WRITE(7,21)((C(I,J), J = 11, 15), I = 1, 17)
WRITE(7,22)((C(I,J), J = 16, 17), I = 1, 17)
CALL EXIT
END
**PROGRAM 5**

**FREQUENCIES OF VIBRATIONS**

```plaintext
REAL*8 A(17,17), B(17,17), C(17,17), D(17,17), E(17,17), F(17,17),
G(17,17), H(17,17), U(17,17), V(17,17), W(17,17), S(17,17), R(17,17),
T(17,17), X(17,17), Y(17,17), Z(17,17), W4(17), W5(17), W6(17)
READ(5,1)(A(I,J), J=1,5, I=1,17)
1 FORMAT(5D15.6)
READ(5,1)(B(I,J), J=6,10, I=1,17)
READ(5,1)(C(I,J), J=11,15, I=1,17)
READ(5,2)(D(I,J), J=16,17, I=1,17)
2 FORMAT(2D15.6)
READ(5,1)(E(I,J), J=1,5, I=1,17)
READ(5,1)(F(I,J), J=6,10, I=1,17)
READ(5,1)(G(I,J), J=11,15, I=1,17)
READ(5,2)(H(I,J), J=16,17, I=1,17)
READ(5,1)(I(I,J), J=1,5, I=1,17)
READ(5,1)(J(I,J), J=6,10, I=1,17)
READ(5,1)(K(I,J), J=11,15, I=1,17)
READ(5,2)(L(I,J), J=16,17, I=1,17)
READ(5,1)(M(I,J), J=1,5, I=1,17)
READ(5,1)(N(I,J), J=6,10, I=1,17)
READ(5,1)(O(I,J), J=11,15, I=1,17)
READ(5,2)(P(I,J), J=16,17, I=1,17)
READ(5,1)(Q(I,J), J=1,5, I=1,17)
READ(5,1)(R(I,J), J=6,10, I=1,17)
READ(5,1)(S(I,J), J=11,15, I=1,17)
READ(5,2)(T(I,J), J=16,17, I=1,17)
Z = 72.36D0
DO 3 I = 1,17
DO 3 J = 1,17
S(I,J) = G(I,J) + H(I,J)
U(I,J) = G(I,J)
V(I,J) = G(I,J)
```

- **DO 3 I = 1,17**
- **DO 3 J = 1,17**
- **S(I,J) = G(I,J) + H(I,J)**
- **U(I,J) = G(I,J)**
- **V(I,J) = G(I,J)**
R(I,J) = S(I,J)
3 T(I,J) = S(I,J)
   CALL DNROOT(17,G,A,AL,X)
   CALL DNROOT(17,U,B,BL,X)
   CALL DNRCOT(17,V,C,CL,X)
   CALL DNROOT(17,S,D,DL,X)
   CALL DNRCOT(17,R,E,EL,X)
   CALL DNROOT(17,T,F,FL,X)
   WRITE(6,4)(I,AL(I),BL(I),CL(I),DL(I),EL(I),FL(I),I=1,17)
4 FORMAT(30H1EIGN VALUES OF SIX PROBLEMS // (I5,6D18.6))
   DO 8 J = 1,15
      I = 16 - J
      W1(I) = DSQRT(AL(J)) / Z
      W2(I) = DSQRT(BL(J)) / Z
      W3(I) = DSQRT(CL(J)) / Z
      W4(I) = DSQRT(DL(J)) / Z
      W5(I) = DSQRT(EL(J)) / Z
      W6(I) = DSQRT(FL(J)) / Z
   WRITE(6,9)(I,W1(I),W2(I),W3(I),W4(I),W5(I),W6(I),I=1,15)
9 FORMAT(40H1 FREQUENCIES OF VIBRATION FOR BENDING ///
   160H IN AIR STRIP TH. WITH A.M. MATRIX /
2/(I4,3F15.5))
   WRITE(6,11)(I,W4(I),W5(I),W6(I),I=1,15)
11 FORMAT(40H1 FREQUENCIES OF VIBRATION FOR BENDING AND SHEAR ///
   160H IN AIR STRIP TH. WITH A.M. MATRIX /
2/(I4,3F15.5))
   CALL EXIT
END
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**Report Title:**
Natural Frequencies of a Body of Revolution Vibrating Transversely in a Fluid

**Author:**
Louis Landweber

**Report Date:**
August 1968

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**Abstract:**
A previously developed procedure, in which the natural frequencies of a body vibrating in a fluid are given by the eigenvalues of the potential energy matrix of an elastic body with respect to the combined kinetic energy matrix of the body and fluid, is applied to a body of revolution. Integral relations between the velocity potential on the body and the vibration velocity, and the expression for the kinetic energy of the fluid as a surface integral are discretized to obtain a quadratic form for the fluid energy. Integrals for the kinetic and potential energies of the elastic body, taking both bending and shear into account, are also discretized. The method is illustrated by application to a form with assumed inertial and elastic characteristics.
Ship vibration
Potential flow
Natural frequencies
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