SOME ASPECTS OF FLOW-INDUCED VIBRATIONS OF HYDRAULIC CONTROL GATES

by
Frederick A. Locher

Sponsored in part by
U.S. Army Corps of Engineers
Waterways Experiment Station
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IIHR Report No. 116

Iowa Institute of Hydraulic Research
The University of Iowa
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CHAPTER I. INTRODUCTION

Abrupt spatial changes in boundary form are common features in many hydraulic structures. Considerable attention has been focused on those cases in which a free shear layer is present in the flow field, since distressing and sometimes catastrophic flow-induced vibrations have resulted from the interaction of the free-shear layer and the structure. Practical examples involving flow-induced vibration include the singing of turbine blades and ship propellers, the galloping of power transmission lines, the buffetting of one of a row of smokestacks, and the vibration of high-head gates in dams and outlet works. Although these illustrations seem unrelated, one factor is common to all: a free shear layer is associated with the flow in each case, and is one of the primary factors in providing a means by which basic flow instabilities ultimately manifest themselves as flow-induced structural vibrations.

The last example, high-head gates in dams and outlet works, has been the principal concern of the present investigation. The geometrical configuration chosen for study (Fig. 1) is a highly idealized representation of a flat-bottomed, high-head gate just protruding into the flow—in effect, a normal wall or step. The primary objectives of this research were to investigate how basic flow instabilities associated with a free shear layer may lead to flow-induced
vibrations of high-head gates, and to relate the mechanisms elucidated by the study of the idealized gate geometry to the general classification of flow-induced excitation presented by Naudascher (1).

A free shear layer forms wherever flow separates from a boundary. When mean flow patterns are constructed, the initial portion of the free shear layer coincides with the mean free streamline. While mean-flow patterns are useful, the time-dependent characteristics of the flow provide the key to the understanding of the phenomenon. Near a separation point, the free shear layer is so thin that it may be considered a surface of discontinuity in the velocity field. Lord Rayleigh has shown that the amplitude of small perturbations in a surface of discontinuity will grow exponentially. Since disturbances from external sources (turbulence in the flow, etc.) are always present in any practical situation, the free shear layer will be disturbed, the amplitude of the perturbations will grow larger, and the layer will roll up into discrete vortices (Fig. 2). The action of the intense shear within the layer in conjunction with downstream convection immediately disrupts and diffuses these vortices, completing the transition to turbulence. In general, then, a free shear layer flow degenerates into the heterogenous eddy motion of turbulence (see path labelled 1 on Fig. 3).

Above a certain Reynolds number, shear flows act as though they possess a selective amplification mechanism, i.e., the amplitudes of disturbances at certain frequencies are amplified, whereas the amplitudes of disturbances at other frequencies are damped. The
A classical example is the phenomenon of Tollmien-Schlichting waves in a boundary layer, the stability criteria for which were predicted theoretically by Tollmien (2), and verified experimentally by Schubauer and Skramstad (3). Thus, if a broad spectrum of random disturbances perturbs a free shear layer, the amplitude of a selected band of frequencies will be amplified more rapidly than those outside the band. From this consideration, as well as from evidence suggested by Fig. 2, one might expect that the resulting fluctuating flow would exhibit a very narrow-band frequency spectrum, and that the boundaries in the vicinity of the shear layer would experience a predominantly periodic excitation. This is indeed the case, provided that some additional control of the energy transfer from the basic flow to the disturbances is present. Without such additional control, transition to turbulence, leading to predominantly aperiodic excitation is inevitable because: 1) both the amplitude and phase of the optimum component of the disturbance are random variables in time, and as a consequence the required phase relationship between the optimum component and the amplification mechanism is not continuously maintained; and 2) other components whose initial amplitudes have a probability of being larger than the amplitudes of the optimum components also compete for the energy being transferred from the shear flow to the disturbances. The presence of these other components interferes with the growth of the optimum component, which is unable to persist long enough to maintain an oscillating flow with a predominant periodicity.
The term control, as used herein, means the introduction or presence of mechanisms by which the disturbance of the free shear layer is so modified and maintained that the resulting flow is oscillatory rather than turbulent (path 2, Fig. 3). It is necessary to distinguish between external controls, such as the forced oscillation of a boundary, and internal controls, which result from the interaction of the flow with its boundaries, particularly since the use of external control as an aid to the understanding of the more complex internal control mechanisms forms the basis of the experimental methods used in the present study.

Internal controls may be most conveniently discussed in terms of a feedback mechanism. After the velocity and pressure fluctuations of the initial disturbances are amplified and convected downstream, they interact with the boundaries of the flow field, producing more disturbances. These disturbances in turn are partially transmitted back to the origin of the free-shear layer and trigger new disturbances, which then proceed through the cycle just outlined. Because of the selective amplification mechanism, the new disturbances result in fluctuations with a much narrower range of frequencies than the original perturbations. With subsequent cycles the fluctuations become more and more nearly periodic. Thus, interaction of the flow with its boundaries coupled with selective amplification forms what may be described as a feedback loop, analogous to the feedback concepts familiar to electrical engineers. The intensity and degree of periodicity of the resulting oscillating component of the flow depends
upon the efficiency of the feedback mechanism and the phase relationship between the generated disturbances and the transfer characteristics of the amplification mechanism.

Internal controls are referred to as self-controls, since the characteristics of flow itself cause the disturbance which ultimately leads to the oscillatory flow. These internal controls may be classified further. If the boundaries with which the flow interacts are rigid, the feedback loop may be either fluid-dynamic or fluid-resonant; if the boundaries are elastic, the feedback loop is called fluid-elastic (refer to Fig. 3). These classifications together with pertinent examples have been discussed by Naudascher (1).

Each of these types of self-control may be illustrated by considering various flow conditions for the geometry depicted in Fig. 1. The ratio of protrusion distance into the flow, b, to conduit width, T, has been maintained at b/T = 1/6, which falls within the range of values where vibration of high-head gates at partial gate openings has been noted by the U.S. Army Corps of Engineers (4). Because the shape of the gate lip is important in determining the vibration characteristics of high-head gates, significant changes in the forces induced on the normal wall would be expected to occur with changes in the ratio of the wall thickness to the protrusion distance, d/b. Separation occurs at the leading edge of the normal wall and forms a free shear layer for all values of d/b. As depicted in Fig. 4, three different flow conditions have been observed (5,6). First, for d/b < 2, the free shear layer remains separated from the normal wall;
second, for $2.5 < d/b < 4.5$, a region of unstable reattachment occurs; and third, for values of $d/b > 4.5$, the reattachment becomes stable.

When $d/b = 1$, the mean surface of separation reattaches itself to the boundary approximately $11.75\ b$ downstream from the leading edge of the normal wall. If this interaction of the flow with the boundary generates disturbances which are transmitted back to the origin of the free shear layer, the existence of a fluid-dynamic feedback loop would be indicated. Other work (5) has shown that the pressure fluctuations measured at $x = 11.75\ b$ do exhibit a "dominant" frequency, but because of the large mass of fluid in the separation zone and the relatively large distance between the point at which the flow interacts with the boundary and the origin of the free shear layer, the disturbances generated apparently do not affect the free shear layer.

As the ratio $d/b$ becomes greater than approximately 2.5, the free shear layer becomes attached to the bottom face of the gate model. Unless the value of $d/b$ is greater than about 4.5, this reattachment is highly unstable, and may be portrayed as shown in Fig. 4b, wherein it is seen that the instantaneous free streamline may, or may not, be attached to the bottom face of the gate model, apparently finding steady equilibrium at neither place. The instability causes the free streamline to oscillate to and fro, alternately being attached to and free from the gate bottom. During this process, the amount of fluid in the regions of separation fluctuates widely. Notice that when the flow is attached to the gate
face, there must be two separation zones: one over the gate and one just downstream from it.

Experimentally, it has been determined that the ratio \( d/b = 3 \) is reasonably close to the value at which the instability attains its maximum intensity (5). Because of the entirely different flow pattern which develops over the range of \( d/b \) characterized by the free streamline instability, a significant change in the characteristics of the pressure fluctuations occurs in comparison with, say, \( d/b = 1 \). The free streamline oscillation results in an increased RMS value of the pressure fluctuations and a predominantly oscillatory flow, as evidenced by spectral analysis of the pressure fluctuations (5) on the bottom face of the gate model (Point B, Fig. 1). One may regard the interaction of the flow with the boundary geometry as a control mechanism which, in part, governs the motion of the free shear layer. Since the disturbances generated by the unstable reattachment are most certainly fed back to the origin of the free shear layer, thus affecting the new disturbances, the phenomenon would appear to be one of self-control involving fluid dynamic feedback, according to Naudascher's scheme of classification (1).

For values of \( d/b \) greater than 4.5 (see Fig. 4), the reattachment becomes stable. Since these geometries resemble forward-facing steps and do not in any way resemble the geometry of high-head gates, the flow characteristics of these cases need not be of concern here.

So far only fixed rigid boundaries have been discussed. No mention has been made of the elastic properties of the structure or
flow boundaries. If the gate model depicted in Fig. 1 is permitted one degree of freedom (i.e., motion in a direction perpendicular to the conduit walls), then one may regard the system as an idealized representation of an elastically suspended gate. With oscillation of the gate model, the flow pattern will be altered significantly, and the feedback loops may be fluid-elastic, fluid-dynamic, fluid-resonant, or any coupled combination thereof. Note that the fluid-elastic feedback loop represents a control mechanism which sustains vibration of the structure only in the presence of structural motion.

It is thus apparent that to initiate flow-induced vibrations, some kind of fluid-dynamic or fluid-resonant feedback is necessary. Once vibration has begun, self-control involving fluid-elastic feedback may result. For the gate model, self-control involving fluid-dynamic feedback exists over the range $2.5 < d/b < 4.5$, a consequence of the flow instability discussed above. A dominant frequency in the pressure fluctuation on the bottom face appears to be of sufficient intensity to excite structural vibrations. The remaining important question with regard to flow-induced vibrations is what effect does displacement of the gate model have on the characteristics of the force induced on the bottom face of the gate. The answer to this question, and the problem of interpreting the results of such a study in terms of the basic mechanisms of self-control, form the basis of this investigation. The experimental techniques, the analysis of the data, and the results of the study are presented in the following chapters.
CHAPTER II. DIMENSIONAL ANALYSIS

Let us consider first a dimensional analysis of flow over a fixed, rigid model gate in which self-control involving only fluid-dynamic or fluid-resonant feedback can occur. In general, the variables describing flow phenomena can be classified into three groups characterizing: 1) the boundary geometry, 2) the fluid properties, and 3) the flow properties. With reference to Fig. 1, the variables describing the boundary geometry are:

\[ T = \text{height of the conduit} \]
\[ L = \text{transverse dimension of the conduit} \]
\[ b = \text{projection of the gate model into the flow} \]
\[ d = \text{thickness of the gate model} \]
\[ x = \text{longitudinal coordinate in the plane of the gate bottom; } X = 0 \text{ at the upstream face of the gate (Fig. 1)} \]
\[ y = \text{transverse coordinate in the plane of the gate bottom wall; } y = 0 \text{ at the upstream face of the wall (Fig. 1).} \]

The fluid properties of significance are \( \rho \) and \( \nu \) where

\[ \rho = \text{fluid density} \]
\[ \nu = \text{fluid kinematic viscosity} \]

The relevant flow properties are:

\[ U_0 = \text{the mean velocity of the approach flow upstream from the gate model} \]
\[ \delta \] = boundary layer thickness of the approach flow

\[ \sqrt{\overline{P'^2}} \] = root mean square value of the fluctuation of pressure from the mean value

\[ \sqrt{\overline{F'^2}} \] = root mean square value of the fluctuation of force from the mean value

\( \Phi \) = the spectral density function of any fluctuating variable, defined such that, for example, \( \overline{P'^2} = \int_{0}^{\infty} \Phi(F) df \)

where \( f \) is frequency.

If the pressure fluctuations at a point are of interest, then an application of the \( \pi \) theorem with \( P, U_o, \) and \( b \) as repeating variables yields the following functional relationship:

\[ \sqrt{\overline{P'^2}} = \frac{1}{\frac{T}{b} U_o^2} \phi_i \left( \frac{b}{T}, \frac{b}{b}, \frac{d}{b}, \frac{s}{b}, \frac{X}{b}, \frac{y}{b}, \overline{\rho} \right) \]

where \( \overline{\rho} \) is the Reynolds number. A general form of the expression is

\[ \sqrt{\overline{P'^2}} = \frac{1}{\frac{T}{b} U_o^2} \phi \left( \alpha_i, \overline{\rho} \right) \]

where the \( \alpha_i \) represent the various geometrical ratios. Similarly the nondimensional spectral density function may be written as

\[ \frac{U}{b} \overline{\Phi \left( \frac{fb}{b} U_o \right)} = \phi_2 \left( \alpha_i, \overline{\rho} \right) \]

For this particular study, both the ratio of wall projection to conduit height, \( b/T \), and the ratio of transverse dimension to wall projection, \( L/b \), were held constant. As long as the boundary-layer thickness is less than the wall projection \( b \), \( U_o \) provides an adequate description of the approach flow (7). Previous measurements (7) have shown that \( \delta \) is approximately 0.72 and remains essentially constant with
the slight variations in Reynolds number used in this study. Therefore the functional relationship for the RMS value of the pressure fluctuations and corresponding spectral density function at a particular point \((x,y)\) on the boundary may be reduced to

\[
\frac{\sqrt{D^2}}{\frac{z}{r}U_c^2}, \frac{U_c}{b} \Phi(f_b) = \Phi_1 \left( \frac{d}{b}, FR \right) \tag{1}
\]

If the dynamic loading, \(F'\), acting on an area, \(A\), (length, \(l\), and width, \(d\)) with centroid \((x,Y)\), on the bottom face of the gate model is considered, then for a particular point \((x,y)\), the reasoning outlined above leads to

\[
\frac{\sqrt{F'^2/A}}{\frac{z}{r}U_c^2}, \frac{U_c}{b} \Phi(f_b) = \Phi_2 \left( \frac{d}{b}, \frac{l}{b}, FR \right) \tag{2}
\]

The simplicity of [1] and [2] is self-evident. An investigation of these functional relationships should not only provide insight into the basic mechanisms responsible for initiating structural vibration, but also should provide a framework for interpreting fluid-elastic effects.

In the final analysis, an engineer faced with a problem involving flow-induced vibrations is interested in the response of the structure to the induced loading. A study of the response or flow-induced displacement of the structure must include the inertial, elastic, and damping characteristics of the structure; these factors greatly increase the complexity of the investigation. Not only is a fourth group of descriptive variables required in addition to the three already considered, but the flow itself is further complicated by an additional path for the transfer of energy—from the flow to the structure, or vice-versa. The following dimensional analysis
considers the response or displacement of a structure with one degree of freedom to forces induced by the flow field surrounding the structure—an analysis relevant to this investigation because the vertical oscillation of high-head gates is motion with essentially one degree of freedom.

Some of the variables in the fourth category which describe the characteristics of the structure are:

\[ m_s \rho_s V_s = \text{the mass of the structure; } \rho_s = \text{density and } V_s = \text{volume of the structure} \]

\[ B_s F_n = \text{"Coulomb" damping resulting from the normal force.} \]

\[ B_1 = \text{the coefficient of kinetic friction} \]

\[ B_2 = \text{coefficient of "solid damping" within the elastic suspension of the structure} \]

\[ c = \text{coefficient of that part of the fluid damping not affected by the flow} \]

\[ k = \text{spring constant of the suspension} \]

\[ q = \text{displacement of the structure} \]

A discussion of a more general nature has been given by Naudascher (20). For purely viscous damping and linear elasticity, a dimensional analysis of motion of the normal wall perpendicular to the direction of the oncoming flow utilizing the fluid density \( \rho \), the reference velocity \( U_0 \), and the mean projection of the wall into the flow \( b \) as repeating variables yields

\[
\frac{\sqrt{\nu}}{b} \triangleq \frac{U}{b} \Phi \left( \frac{f b}{U_0} \right) = \phi_{5,6}\left( \frac{\rho U_0}{b} \frac{U_0^2}{c^2 b}, \frac{B_s}{\rho U_0^2 b}, \frac{B_k}{c^2 b}, \frac{k_1}{c^2 b}, \alpha_{1,1} \right) \tag{3}
\]
Note carefully that any study of structural displacements thus involves the elastic, damping, and inertial characteristics of the structure, which must be scaled according to the applicable laws of similitude (the Reynolds criterion in this case). The construction of a dynamically similar model is a difficult (if not practically impossible) task, particularly if the Froude, Mach, Weber, or cavitation number must be considered in addition to the Reynolds number. Slight changes in the inertial, elastic, or damping characteristics of the model may result in significant changes in the characteristics of the measured displacements, and hence misleading prediction of prototype performance. Furthermore, only the most complex and expensive structures would justify the cost of a study sufficiently extensive to define the functional relationship given by [3].

At the present state of the art, it is advantageous to exclude consideration of the structural parameters, particularly in studies of a general nature, and to investigate displacement effects by another method. The most straightforward approach is the forced vibration of a rigid, low-inertia model at a series of known amplitudes $a_0$, and frequencies $f_0$. Such a procedure changes the focus of the investigation from flow-induced displacements to flow-induced forces, since the displacement now becomes an independent rather than dependent variable. The functional relationship for the dynamic force induced on an area, $A$, on the bottom face of the normal wall is similar to the expression for the flow-induced displacement [3] above,
\[ \sqrt{\frac{F^{*2}}{A}} \frac{U_b}{b} \Phi \left( \frac{f_b}{U_b} \right) = \phi_{_3}(\frac{a_0}{b},\frac{f_0 b}{U_b},\frac{C}{\rho U_b^2},\frac{B_0}{\frac{1}{2} \rho U_b^2 b},\frac{B_2}{\frac{1}{2} \rho U_b^2 b},\frac{C}{\frac{1}{2} \rho U_b^2 b},\frac{k}{\frac{1}{2} \rho U_b^2 b}) \] [4]

With forced vibration of the normal wall in the y direction, the corresponding functional relationship is

\[ \sqrt{\frac{F^{*2}}{A}} \frac{U_b}{b} \Phi \left( \frac{f_b}{U_b} \right) = \phi_{_{2,0}}(\frac{a_0}{b},\frac{f_0 b}{U_b},\alpha_i,\mathcal{R}) \] [5]

It is self-evident that the experimental program needed to define \( \phi_{_{2,0}} \), given by [5] is much more tractable than that needed to obtain \( \phi_{_{2,0}} \), appearing in [4]. To be sure, significant connections in the various feedback loops in the fluid-elastic system have been modified or eliminated, and the results of such a study must be interpreted accordingly. Nevertheless, the study of flow-induced loading suggested by [2] and [5], being independent of specific structural characteristics, appears more suitable for determining the general nature of fluid-elastic effects than would a study of flow-induced displacements. The more general results of an investigation of flow-induced loading would therefore be useful as criteria in preliminary design stages, where many alternative possibilities might be considered, and invaluable in determining which of the many dynamic characteristics of the structure must be simulated in a model should further studies be contemplated.

Finally, it should be noted that at the relatively large values of \( \mathcal{R} \) occurring in prototype structures and used in this study, the vibrational behavior would be expected to be independent of \( \mathcal{R} \).
Cavitation

Although an investigation of cavitation effects was not the primary objective of this study, the influence of cavitation on flow-induced vibrations certainly cannot be neglected or disregarded. If the vapor pressure is added to the list of flow properties, the functional relationship for the RMS value of the pressure fluctuations and the spectral density function at a particular point on the bottom face of the normal wall are

\[
\frac{\sqrt{F^{12}}}{\frac{1}{2} \rho U_0^2} \frac{U_0}{b} \Phi \left( \frac{fb}{U_0} \right) = \phi_n \left( \frac{d}{b}, \frac{\rho}{\rho_0}, K \right)
\]

where \( K \) is a cavitation parameter defined as \( \frac{p_0 - p_v}{\frac{1}{2} \rho U_0^2} \). Herein, \( p_0 \) is the reference pressure in the uniform flow upstream from the normal wall, and \( p_v \) is the vapor pressure of the fluid. The functional relationship corresponding to [2] is

\[
\frac{\sqrt{F^{12}}/A}{\frac{1}{2} \rho U_0^2} \frac{U_0}{b} \Phi \left( \frac{fb}{U_0} \right) = \phi_{12} \left( \frac{d}{b}, \frac{\rho}{\rho_0}, \frac{\rho}{\rho_0}, K \right)
\]

The significance of cavitation in flow over a normal wall is discussed in later chapters.
CHAPTER III. EXPERIMENTAL APPARATUS AND INSTRUMENTATION,
VIBRATING GATE MODEL

Water Tunnel

The experimental work for this study was carried out in a closed-circuit water tunnel located in the laboratory of the Iowa Institute of Hydraulic Research. This water tunnel has a rectangular test section measuring approximately 6 inches wide, 24 inches deep, and 36 inches long. A 65 HP, 3600 RPM motor and mechanical speed reducer (65:1 ratio) coupled to a fluid drive and impeller permitted a continuous variation of fluid velocity in the working section up to a maximum of 36 feet per second (Fig. 5).

An auxiliary side tank connected either to a vacuum pump or a pressure pump allowed a variation of ambient pressure from -9 psig to 20 psig, measured at the low-velocity region immediately upstream from the test section with a simple mercury U-tube manometer. Control of the cavitation number over a wide range was thus provided.

Gate-Model Geometry

Earlier experiments (7) showed that corner vortices at the junction of the gate model and the top and bottom of the test section can seriously influence the entire flow pattern if the lateral dimension of the gate model is too small in relation to its projection
into the flow. Because the water-tunnel test section is only 6 inches wide, the gate model was mounted vertically, providing a ratio of \( L/b = 24 \). It was felt that the effects of the corner vortices on measurements made near the centerline of the gate model were negligibly small (5).

The gate model was designed so that spacer blocks could be inserted to provide expansion from a geometric ratio of \( d/b = 1 \) to \( d/b = 3 \). All of the main frame for \( d/b = 1 \) was also used for the gate with \( d/b = 3 \).

Rather than using a pressure transducer to obtain measurements of the flow-induced forces, as Tatinciaux had done (5), a direct measurement was made of the fluctuating force on the bottom face of the gate model. Several plates of various lengths and spanning practically the entire thickness of the gate, \( d \), were constructed (refer to Fig. 6). A change of the length, \( L \), of the plates required that the entire apparatus be removed from the water tunnel. By proper spacing of the bolt holes, any of four different plate lengths, \( L/b = 1, 2, 4 \), and 6, could easily be attached to the force transducer. Calibration of the transducer setup was necessary after each change of plates.

One of the most difficult problems encountered was the construction and maintenance of a watertight seal around the vibrating gate model. The vibrating section of the gate model and the slot in the back plate of the water tunnel are both rectangular. It is difficult to obtain a material which will stand up under repeated stress while
maintaining the double-curved surfaces necessary to form the corners. Ordinary rubber developed cracks and subsequent leaks after a short period of time. The most satisfactory material tested was CHORlastic 9255 silicone rubber, a flexible, tear-resistant synthetic available in 1/32-inch thickness. The joint between the end of the vibrating wall and the stationary backplate was particularly difficult to seal. Here, Silastic bathtub caulk manufactured by Dow-Corning was used to provide a more positive seal. The bathtub caulk is really a room temperature vulcanizing compound, and cures to a flexible substance which adheres to the silicone rubber, brass, or steel with about equal tenacity. This combination of gasket and caulking proved to be an excellent water seal, and permitted vibration of the gate model with little resistance.

**Force Transducer**

Another major problem was finding a transducer which is insensitive to eccentric loading. Apparently most force transducers are constructed under the assumption that the applied load will be concentric (or nearly so) with the axis of the transducer itself. Eccentric loadings, which may be resolved into an axial load and a couple, cause erroneous readings because of the moment induced by the couple. A dynamometer constructed by Newsham (8) and modified slightly for use of a strain-gage force cell was found to be moment insensitive, but too cumbersome to use in this case. Satisfactory performance of a piezo-electric quartz pressure transducer in previous work (6,9), used for the measurement of the fluctuating pressure, led to the
adoption of a piezo-electric load washer, Model 901A, manufactured by Kistler Instrument Corporation. This unit has several quartz crystals around the periphery of the load washer, and the addition of the signals within the transducer is such that the effect of an eccentric load cancels out. In other words, if a plate is attached to the load washer (Fig. 6), the transducer yields the same output no matter where the load is applied on the plate. The load washer was found in static tests to be moment insensitive for eccentric loads of 1/2 Kg positioned 6 inches from the axis of the transducer. It is quite possible that eccentric loads approaching the full scale capabilities of the load cell could not be measured satisfactorily. A few words of caution are in order, however.

First, the transducer must be preloaded by prestressing the load washer between two surfaces, as shown in Fig. 6. Unfortunately, the insensitivity to moments is a function of one's technique in pre-loading, and a rather frustrating series of trials must be made before the moment insensitivity is obtained. It was found that preloading to a higher load than required, and then backing off slightly seemed to be the most efficient method for producing the desired results. Second, piezo-electric transducers are suitable for fluctuating measurements only. Although the cell will hold mean values for a short time (usually a matter of seconds) which permits static calibration, it is not possible, in general, to use them for actual measurement of mean values (except under unusual circumstances). Third, piezo-electric transducers are sensitive to accelerations. Thus, some
form of acceleration compensation is necessary unless the inertial load applied to the load cell is very small; this point is considered at length in a subsequent section. The mass of the largest plate and support used in this study was small enough that vibrations transmitted from the water tunnel motor, drive, and pressure pump produced negligible transducer output.

Because this study was concerned only with the characteristics of the fluctuating force on the flat bottom of the gate model, and because the magnitude of the induced fluctuations was small, the load cell proved to be entirely adequate. The small size of the cell was particularly advantageous. In retrospect, however, a moment-insensitive strain-gage transducer would have been preferable. The calibration procedure would have been simpler and mean values could have been obtained if desired.

**Transducer Mounting**

Although the problem of moment sensitivity was solved through proper choice of transducer, the correct mounting of the load cell in the water tunnel was not so easily accomplished. The transducer was mounted in the normal wall as shown in Fig. 6. A clearance of about 0.005 inches separated the plate from the rest of the gate, and the flexible gasket made of Cordet sheet packing maintained a watertight seal. A great deal of care was necessary in mounting the transducer in order to preserve the required moment insensitivity. The gasket was the principal source of trouble.
If the gasket was stretched or compressed too much at the edges by the restraints, stresses occurred which resulted in spurious readings for eccentric loads. It was determined that if the gasket were wetted, the material became even more flexible, the unwanted stresses were relieved, and insensitivity to eccentric loading was restored. The gasket was thoroughly wetted when the apparatus was installed in the water tunnel, and the arrangement then proved to be satisfactory. Calibration was carried out with a dry gasket.

Changes in the size of the plate attached to the force transducer were accomplished as quickly as possible. As long as the Cordet paper gasket did not completely dry out, the calibration curve for the force transducer remained essentially unchanged. When the gasket dried out, it shrank and was no longer useable. In future studies, it is recommended that a more flexible gasket material be selected, one which does not change its dimensions during successive wet and dry cycles.

It should be mentioned that end effects did occur. That is, some load was transmitted through the Cordet paper gasket at the extreme ends of the plate. When a load was placed as close to the sensing plate as possible, tests indicated that about 2% of the load was transmitted through the dry gasket to the transducer-mounted plate. For a wet gasket even less than 2% was transmitted, but it should be noted that the effects of dynamic loading while in operation are unknown.
Vibration Apparatus

Variation in the amplitude and frequency of the forced oscillations was accomplished through the use of commercially available equipment. A 1 HP motor and a SCR speed control with a 1% speed regulation supplied by the Boston Gear Works (Quincy, Massachusetts) were used to drive the two crank heads shown in Fig. 7. The adjustable cranks were obtained from The Budd Company (Phoenixville, Pennsylvania), and are components from a 5000 lb fatigue-testing machine.

The crank arms were guided by slide bearings to insure rectilinear motion. Stainless steel push rods passing through oil-impregnated bronze bearings connected the crank arms to the normal wall (Fig. 7). The drag force on the normal wall resulted in a slight "cocking" action in the bearings, which, in turn, produced undesirable chatter in these bearings. Springs were provided to counteract this drag force. With proper adjustment, satisfactory operation was attained. The remaining friction in the bearings was troublesome, and some bearing chatter could be observed on the output of the load cell. The extraneous vibrations were small, but had there been more massive plates attached to the load cell, this noise soon would have become intolerable. With benefit of hindsight, it is realized that a framework of flexible steel plates rather than guide blocks and sleeve bearings would have been much more suitable, and probably would have eliminated the problems associated with alignment, bearing chatter, and bearing friction. In general, however, the mechanical aspects
of the design were adequate, and its performance was altogether satisfactory.

**Instrumentation**

A Kistler Model 901A Load Washer and Model 566 Charge Amplifier were selected to measure the fluctuating forces applied to the plate representing the bottom of the prototype gate. In this transducer, quartz is used as the piezo-electric element. A force applied to a properly oriented elastic quartz crystal causes a change in electric charge of the quartz element which is directly proportional to the applied load, a phenomenon known as the piezo-electric effect. The change in electrical charge is sensed and amplified through by a charge amplifier, and an output voltage proportional to the applied force is provided.

The forced oscillation of the gate model introduced the problems of acceleration compensation and virtual mass effects. The acceleration of the base to which the transducer was attached and the mass of the plates supported by the transducer induces a force on the transducer. Furthermore, an unknown mass of fluid (the virtual, or added mass) is also accelerated, resulting in another induced force. Both of these forces must be subtracted from the output of the load cell in order to obtain a signal which represents only the flow-induced forces.

At first, an accelerometer was attached to the vibratory apparatus in an attempt to compensate for acceleration and virtual mass. High frequency noise from bearing chatter rendered this approach
unsatisfactory. The difficulty arose from the fact that, considering sinusoidal displacements, the acceleration is proportional to the square of the frequency, so that the low-frequency, forced oscillation (0 to 10Hz) becomes distorted by the high frequency "noise."

This problem was overcome by using a linear displacement potentiometer which supplies a voltage proportional to the displacement of the gate model itself. Suitable electronic circuitry for the subtraction of the displacement signal from the output of the load washer was designed and built by Dr. J. R. Glover of the Institute Staff. The operating procedure consisted of simply vibrating the gate model at a specific frequency in still fluid and adjusting a potentiometer until the signal from the load cell and the displacement gage essentially cancelled. Fig. 8 shows the result of one such operation. The small disturbances on both oscilloscope traces represent the effects of bearing chatter and vibration of the drive components.

There is no question that the use of a voltage proportional to the displacement of the gate model to compensate for inertial effects is an adequate measure. Furthermore, compensation for the virtual mass in quiescent fluid can be successfully accomplished electronically as is demonstrated by the results shown in Fig. 8. With the fluid in motion, however, there is apparently a phase shift at the larger amplitude ratios and higher frequencies between the displacement and the force induced by the virtual mass. When the gate was vibrated at the larger amplitudes and higher frequencies, a significant volume of
fluid was displaced (up to 2 3/4 cubic inches for d/b = 3 and a₀/b = 0.03), and the entire water tunnel was pulsed with a pressure fluctuation at the forcing frequency. This oscillation was quite visible in the mercury U-tube manometer connected to the water tunnel upstream from the gate model.

The effects of virtual mass are important. If the virtual mass is different for still and moving fluid (as is most probably the case), then the amplitude of the compensating signal will be incorrect. If a phase shift occurs, the electronic compensation technique is invalid, and actual addition of the signals may occur. The consequences of these inadequacies in acceleration compensation will be discussed in context for each of the geometrical forms investigated, d/b = 1 and d/b = 3.

An entirely different approach lies in a technique known as signal averaging (10). If it is known that a periodic waveform is present, but masked within a "noisy" signal, this waveform may be reconstructed by signal averaging. A signal with precisely the same frequency as the desired waveform is necessary for synchronization of the procedure. The "sync" signal need not be in phase with the repetitive waveform, and hence the voltage from the linear motion potentiometer would be suitable. A computer might be used to correct the spectrum or RMS values. Regrettably, lack of time precluded application of this relatively new technique in this investigation.
CHAPTER IV. PRELIMINARY INVESTIGATIONS AND EXPERIMENTAL PROCEDURE

Preliminary Investigation

With respect to this study, the most relevant work carried out previously at the IIHR is that of Tatinclaux (5). In his work, the pressure fluctuations were measured at a point on the bottom of the gate model by means of a pressure transducer. Since the completion of Tatinclaux's study, two important improvements in equipment and data analysis have become available. First, a much smaller pressure transducer has been developed. Tatinclaux's pressure transducing system, patterned after that of Newsham (11) and Locher (7), had undesirable frequency-response characteristics if the slightest amount of air was present in the system. This problem is discussed elsewhere (5,11). A flush-mounted transducer used by Chu (8) practically eliminated these difficulties. Second, an IBM 1801 digital computer, which permits much more reliable processing of the experimental data, was recently installed in the Institute laboratory.

In order to make valid comparison between Tatinclaux's work and the present study, it was necessary to repeat some of his measurements using the flush-mounted transducer and to extend his work to include the effects of cavitation. These experiments were conducted in the water tunnel described in the preceding chapter.

A Kistler Model 601A pressure transducer was installed on the bottom face of the normal wall at Point B in Fig. 1. The spectral
density function was obtained with the analog equipment described by Tatinclaux (5), Locher (7), and Chu (9). These analog spectra agreed with Tatinclaux's. It may therefore be concluded that with proper care the piezometric system as used in his work yields reliable measurements of the fluctuating pressure. Data were also obtained to define the variation of the relative RMS value of the pressure fluctuations with cavitation number for \( d/b = 1 \) and \( d/b = 3 \).

**Spectral Density Measurement**

The Institute's IBM 1801 digital computer is equipped with a special-purpose analog package which permits the controlled sampling of any analog voltage within the limits of \( \pm 5 \) volts. An analog to digital converter (ADC) makes digital processing of the analog voltage feasible. This system is described in detail by Glover (12). Considerable time and effort was spent developing a program to perform an auto-correlation analysis of the incoming analog signal. The spectral density function may be obtained from the auto-correlation function through application of a Fourier-cosine transformation. Discussion of this technique is given in Appendix A. Suffice it to mention here that the program was developed, is statistically reliable, functions properly, and produces an auto-correlation function from which the spectral density can be obtained. The details of the program, the choice of the important parameters, and a flow chart are presented in Appendix B.
The logical next step was to use the computer program in conjunction with the flush-mounted pressure transducer and compare the results of the analog technique with those obtained with the on-line digital analysis. This comparison proved both interesting and informative. As was anticipated (5) and as has been proved by Glover (13), there are certain errors inherent in the analog technique. This subject has been discussed thoroughly by Glover (13). The principal conclusion is that the digital technique is more reliable and more accurate, particularly in the low frequency range. Since the vibration of large massive structures is usually at very low frequencies (0 to 10 Hz.), and since the most significant aspects of this study are also of low frequency character, this conclusion is important.

Spectral density functions of the pressure fluctuations were obtained at Point B utilizing the flush-mounted pressure transducer and digital computer for both $d/b = 1$ and $d/b = 3$. The significance of these data is discussed in connection with the presentation of results obtained using the force transducer. Discussion of the data analysis has been relegated to the appendices because a complete treatment at this point adds little to the understanding of flow-induced structural vibration.

**Experimental Procedure**

Since the quantities measured during the course of this study were random variables, considerable care was needed in the determination of all the relevant statistical averages. For example, when the RMS value of the fluctuating force as a function of cavitation
number was required, the following procedure was applied. First, the water in the tunnel was deaerated by running the system for fifteen minutes at a low pressure producing severe cavitation. The dissolved air was thereby brought out of solution and transported to the high points in the tunnel. When the flow was stopped and the pressure returned to atmospheric, the air could be bled off. This sequence was repeated at least three times, until the cavitation had a sharp, crisp sound indicating that the cushioning effect of the dissolved air had been minimized. The water tunnel was operated under various combinations of velocity and pressure to vary the cavitation number over a wide range.

A computer program was written to evaluate the mean square of the random voltage representing either the pressure or force fluctuation. The ADC sampled the incoming voltage at approximately 1485 samples per second. An averaging time of 136 seconds was used, yielding about 202,000 samples for each run. For a given cavitation number, three 136-second averages were obtained. These three runs were averaged; the result was reduced and plotted. Several sets of data were obtained on different days, and if the sets of data superposed, it was assumed that the data were representative. Three separate averages were used, rather than one 408-second average, for the following two reasons:

1. The three values usually were close to each other, indicating that the averaging time is sufficiently long, and
2. If something goes wrong with the electronic system during a run, only a little more than two minutes, rather than six minutes, elapses before the malfunction becomes obvious. The use of the computer for such a simple analysis admittedly is overpowering the problem. In the long run, however, considerable time is saved. The circuit used in earlier work (5,7,9) required frequent calibration and adjustment. The computer, on the other hand, required no adjustment and produced more accurate results than were previously obtainable.

A similar procedure was adopted in determining the RMS value of the fluctuating force for the studies involving forced vibration of the gate model. The model was vibrated in still fluid and the circuit adjusted to compensate for the effects of acceleration and virtual mass. Usually one adjustment was sufficient for a small range of frequencies. The velocity of the fluid in the tunnel was adjusted to a predetermined value, and the RMS value of the fluctuations was measured three times, averaged, and plotted. Three or more separate sets of data were obtained on different days. If the various data agreed satisfactorily, the results were accepted.

Auto-correlation analyses yielding spectral density functions were obtained at particular points of interest. Estimates of the spectral density function in themselves form a random process. The statistics of this process must be examined to insure reliability in
the estimates (refer to Appendix A). Hence two separate analyses were programmed in order to guarantee repeatability and to make certain that the results were not influenced by vagaries in the electronics. For some conditions three or more spectra were obtained. The use of the digital computer significantly reduces the time required for spectral analyses as compared with the analog procedures, and allows a more complete and reliable treatment of the aspects of flow-induced vibration. A description of the advantages of digital analysis may be found in Appendix B.
CHAPTER V. PRESENTATION AND DISCUSSION OF RESULTS

In all experiments the ratio b/T (see Fig. 1) was maintained constant at a value of 1/6; i.e., the projection of the gate model, b, into the flow was one inch throughout the study. Two different gate widths, d, were investigated: d/b = 1 and d/b = 3. The experimental results were obtained at Reynolds numbers,

\[ \frac{IR}{U_0} = \frac{U_0 T c}{\mu} \]

in the range 1.1 to 1.4 \( \times 10^6 \), corresponding to mean flow velocities \( U_0 \) of approximately 21 to 24 fps. Spectral density functions obtained over this range of Reynolds numbers all superposed when plotted in nondimensional form.

The subsequent discussion is organized as follows: for d/b = 1 and the gate model stationary, the results obtained with a pressure transducer are evaluated and considered in the light of the spatial nonstationarity of the pressure fluctuations on the bottom face of the gate model. These results are then compared with those obtained from measurements of the fluctuating force using the force transducer and plates of varying length. All of the data for cavitating flow were obtained with the gate model at rest; these cavitating flow data are compared with the non-cavitating flow data. The results of the forced oscillation of the gate model with d/b = 1 are then discussed in detail.
For d/b = 3, a similar outline has been followed. First, the data obtained with a pressure transducer (gate stationary) are compared with the data for l/b = 6, the only plate length used in this phase of the study. Then the results for cavitating flow (again with the gate stationary) are compared with the results of the non-cavitating flow and contrasted with the results for d/b = 1. Finally, the experimental data for forced oscillation of the gate model with d/b = 3 are presented and comparisons are made with results previously obtained during this study.

Gate Model Stationary: d/b = 1

One of the principal limitations in the practical utilization of the data from the pressure transducer lies in the fact that the transducer provides essentially a point measurement. With a single transducer, therefore, no information regarding the correlation of the pressure fluctuations from point to point in space can be obtained. The precise details of this spatial correlation need not concern us here, but their overall effect is of consequence, especially with regard to the physical size of the transducer.

To simplify the discussion, consider a flat, smooth surface on which the fluctuations are stationary in space; i.e., the statistics of the pressure fluctuations remain constant and independent of the transducer position. A pressure transducer with a finite-size sensing area can only resolve pressure fluctuations with a frequency less than some upper limit which is a function of both the scale of the turbulence in the flow and a characteristic dimension of the
transducer face. As an individual eddy passes a given point, a fluctuation in pressure is recorded. The small eddies give rise to high-frequency fluctuations. Some eddies will be so small that a finite size transducer cannot resolve the fluctuations they produce. Therefore, as the size of the transducer face is increased, fewer high-frequency fluctuations will be observed, the RMS value of the pressure fluctuations becomes smaller, and the estimates of the high-frequency constituents of the spectral density function will be erroneous. There has been considerable effort (14,15,16) expended in estimating the effects of finite transducer size, particularly in boundary-layer work.

The experimental arrangement of the force transducer used in this study is equivalent to a pressure cell whose sensing area is variable. As the length of the plate used to measure the fluctuating force on the face of the gate model was varied, both the length of the plate and the spatial nonstationarity of the pressure fluctuations on the surface of the plate had to be taken into account in the interpretation of the experimental results.

Let us now consider some aspects of the lateral correlation of the pressure fluctuations. As the free shear layer generated at the leading edge of the gate model begins to break down, the layer rolls up and forms discrete vortices, the axes of which are oriented primarily parallel to the gate (laterally). The interaction of the intense shear in conjunction with convection of the vortices downstream causes a stretching and breakup of the vortices. The
orientation of the axes of the vortices then shifts toward the longitudinal (downstream) direction. It is reasonable to visualize an instantaneous cellular structure, such as has been observed on circular cylinders (17), for example, in the lateral direction. Therefore, in order to ensure measurements on the face of a two-dimensional structure which are truly representative of the flow-induced forces, the lateral dimension, L, of the plate must be larger than some characteristic length of the cellular flow pattern. Little is known about the spanwise correlation of the separated flow under investigation. Hence, a series of experiments were performed with different plate lengths in order to learn more about the lateral correlation, and to determine the plate length required to yield representative measurements for design purposes.

One may reason that as the length of the plate is increased, the correlation between the pressure fluctuations at the center of the plate and points at either end of the plate will decrease, finally attaining constant value. Once the correlation ceases to be a function of plate length, the forces induced upon the plate may be considered truly representative of the flow-induced forces and not dependent upon the location or size of the transducing element. Because of characteristics of the spatial correlation of the pressure fluctuations, the RMS value of the fluctuating force per unit area should be larger for the smaller plate lengths, decrease with increasing plate length, and reach a constant value as the correlation becomes independent of plate length. Thus, as shown in Fig. 9, the
value of the relative RMS value of the fluctuating force per unit area decreases with increasing plate length, attaining a constant value (within experimental uncertainties) near \( l/b = 6 \). For reference purposes, a point indicating the data from the pressure cell located at Point B on the face of the wall has also been included. It should be borne in mind that the face of this transducer does not extend over the full width of the face of the normal wall. However, the variation in RMS force per unit area with plate length is evidence that the spatial correlation and its effect on transducer size must be taken into account in interpreting the experimental results. The plate lengths chosen for \( d/b = 1 \) were 1", 2", 4", and 6", or equivalently, \( l/b \) ratios of 1, 2, 4, and 6. For each of the plates, the variation of the relative RMS values of the fluctuating force induced upon the plate was also determined as a function of cavitation number. Spectral density functions of the fluctuating force were also obtained for non-cavitating conditions. These results are presented in a subsequent section of this chapter.

**Cavitation Flow.** The data obtained with the force transducer for \( l/b = 1, 2, 4, \) and 6 (see Figs. 12-15), and the pressure transducer located at Point B (5) both show the same trends as the cavitation number is varied. For non-cavitating flows \((K > 4)\), the RMS values are unaffected by changes in the absolute pressure, as was also noted by Newsham (11), and as would be expected from physical considerations. With the onset of cavitation, the value of the relative RMS fluctuations increases slowly, then attains a maximum near \( K = 2.22 \), and
then abruptly decreases (Figs. 12-15). The initial rise most probably reflects the acoustic noise generated during incipient cavitation. The sound waves are transmitted through the fluid to either the pressure cell or the plate. The output voltage from the transducer acquired a "fuzzy" appearance resulting from the high-frequency fluctuations induced by the cavitation. However, the significant peak in the RMS data cannot be completely explained by an increase in intensity of the pressure fluctuations emanating from the acoustical phenomenon associated with cavitation, as the following results demonstrate.

Spectral analysis of the pressure fluctuations obtained from the pressure cell flush-mounted at Point B with $K = 2.22$ shows a definite increase in the low-frequency range of the spectrum when compared with similar data for non-cavitating flow (Fig. 10). Other investigators (18,19) have found that cavitation appears to stabilize the eddies generated in free shear layer flows against breakdown. With a decrease in pressure load on the system, the cores of the eddies in the flow cavitate first, and the vapor cavities stabilize these eddies. It is reasonable to postulate that the stabilizing influence is most pronounced on those eddies containing most of the turbulence energy. Consequently, the persistence of the amplitude of those frequencies associated with the dominant eddy structure (or its effects) is enhanced by the cavitation, giving rise to the increased RMS values and increase in the low-frequency content of the spectral density.
Whether the increase in the low-frequency range of the spectrum of the pressure fluctuations for \( K = 2.22 \) as compared with the spectrum of the non-cavitating flow is a consequence of cavitation alone, or is the result of increased fluid-dynamic feedback within the separation zone behind the normal wall, is uncertain. Earlier work (5) has shown that there is a dominant frequency associated with fluctuations of the instantaneous stagnation point at the end of the separation pocket. Furthermore, it is also clear that this disturbance is transmitted upstream. It seems reasonable to conclude that cavitation stabilizes eddies, modifies the mechanism of fluid-dynamic feedback within the separation zone, and thus causes an increase in the low-frequency range of the spectrum.

Although the general trends of all the data for the various plate lengths used in this study are the same (Figs. 12-15), the differences in detail are also informative. If the hypothesis that cavitation stabilizes the dominant eddy structure against further breakdown is correct, then it is reasonable to assume that the lateral correlation is similarly enhanced. Quite probably the lateral scale would not be changed greatly, but the organization within a distance of the order of the lateral scale would be considerably augmented. Consequently, for cavitating flow the transducer faces would register an increased RMS value of the fluctuating force in comparison with the values for non-cavitating flow.

Figure 11 is a plot of the relative RMS value of the fluctuating force as a function of \( l/b \) with cavitation number as the third
parameter. Except for \( \ell/b = 4 \), the results are consistent. For this case, the values for \( K = 3 \) and \( K = 6 \) are less than the corresponding values for \( \ell/b = 6 \). Troublesome gasket leaks were experienced with cavitating flow for \( \ell/b = 4 \). The results for cavitating flow with \( \ell/b = 4 \) are therefore questionable, particularly in view of the fact that during the experiments for \( \ell/b = 6 \) the apparatus was dismantled, reassembled, and recalibrated, and consistent, repeatable results were obtained. Air cushions the cavitation as well as relieving the pressure behind the gate model. Experience with an air leak for \( \ell/b = 1 \) (subsequently eliminated) showed that significantly lower values for cavitating conditions were found if there were air leaks in the system.

Figure 11 shows that incipient, or slight cavitation (\( K = 3 \)), increases the RMS value of the fluctuating force in comparison with the values for no cavitation (\( K = 6 \)). The relative intensity of the low-frequency components has been augmented as shown by the spectral density functions (Fig. 10). Since the eddies formed during the initial breakdown of the free shear layer are in the low-frequency range of the spectrum, the change in the spectral density function and the increase in the RMS value of the fluctuating force appear to be the result of increased lateral correlation of the pressure fluctuations caused by the stabilizing influence of cavitation.

One further effect, attributable to the fact that the gate model is mounted vertically in the tunnel, should be mentioned. As \( \ell/b \) increases, the peak in the cavitation data (shown in Figs. 12-15) near \( K = 2.22 \) broadens. The cavitation number reported herein refers to
the pressure at the centerline of the tunnel test section. Since
the gate model is mounted vertically, the absolute pressure is not
constant along the bottom of the gate. The value of the cavitation
number therefore varies slightly from the lower to the upper end,
with the effects being more pronounced with the longer plates.
Hence the upper end of the longer plate experiences cavitation
before the centerline, and the effects of cavitation therefore
appear at lower values of the cavitation number at the centerline.
The peaks broaden, and shift slightly toward lower values of the
cavitation number, as may be seen in a close inspection of Figs.
12-15.

Spectral Density. Spectral density functions of the fluctuating
force for non-cavitating conditions were also determined and are
presented in Figs. 16-19. It should be emphasized that plotted in
the nondimensional form shown, the spectral density function indicates
the relative intensity of the fluctuations per unit of circular
frequency. Thus, consistent with the discussion of the effects of
transducer size, the relative magnitude of the low-frequency content
of the spectrum increases as $L/b$ increases, while the relative inten-
sity of the high-frequency range decreases. Because of the vertical
scale chosen in Figs. 16-19, the latter part of this statement is not
evident. A close examination of the data does, however, substantiate
this fact. As was evident from the results obtained from the pressure
transducer, it is clear that there is no significant dominant frequency
present.
Forced Oscillation of the Gate Model: \( d/b = 1 \)

One of the interesting facets of this geometrical configuration is that the location of the separation point is fixed, unlike flow past the more familiar circular cylinder. The free-shear layer as a whole is therefore more sensitive to oscillation of the structure. Other investigators (19) have oscillated sharp-edged plates normal to the direction of flow and have observed that strong vortices are generated with the same frequency as the forced oscillation. Since the mean surface of separation appears as qualitatively depicted in Fig. 4a, and is not yet influenced appreciably by the gate geometry, much of the flow field can be affected by perturbation in the free-shear layer. Hence, it was not surprising to observe a marked change in the characteristics of the fluctuating force once oscillation of the model gate was initiated.

As may be concluded from the spectral density function shown in Fig. 20, most of the fluctuations are concentrated at the frequency of the forced oscillation. The negative values on either side of the large peak result from the smoothing of the data and the "spectral window" discussed in Appendix A. The fluctuating force was completely dominated by the frequency of the forced oscillation throughout the range of frequencies tested, 0 to 15 Hz. A significant increase in the relative RMS value of the fluctuating force was recorded. This change may be explained as follows: as depicted in Fig. 4a, the mean surface of separation and hence the flow field are not appreciably influenced by the boundary form at \( d/b = 1 \). With no forced
oscillation, the fluctuation originating in the free shear layer must be transmitted through the relatively quiescent fluid of that portion of the separation zone between the bottom surface of length \( d \) and the free-shear layer. The breakdown of the free shear layer into the intense eddy structure of turbulence thus occurs above and away from the bottom surface of the gate. Much of the pressure fluctuation is not sensed, in part because of the distance between the source of the fluctuations and the gate, and in part because of the intervening zone of stagnant fluid.

On the other hand, once oscillation of the model gate is initiated, a strong vortex is immediately formed at the leading edge of the model. This vortex breaks down, to be sure, but its immediate effect is to control the oscillation of the free shear layer and to bring the relatively stagnant fluid immediately above the bottom surface of the gate into oscillatory motion. Thus, an oscillating pressure field is set up with a frequency corresponding to the frequency of forced vibration. A significant increase in amplitude of the fluctuating force is the consequence of the mass of fluid in the immediate vicinity of the plate being brought into motion. The output of the force transducer is almost sinusoidal because of the forced vibration controlling the free shear layer. Random pressure fluctuations are still present, but are almost swamped by the periodic pulsations which are produced.

The variation of the relative RMS value of the fluctuating force with dimensionless forcing frequency is depicted in Fig. 21. Note
the approximately three-fold increase in RMS value from the stationary to the oscillatory case. Although there is some scatter, all three sets of data indicate a slight rise in the range $0.03 < fb/U_0 < 0.04$, and a more or less constant level between $fb/U_0 = 0.04$ and $0.058$. The last point for all three sets dropped slightly. One might expect that problems resulting from mechanical vibrations of the experimental apparatus, and the magnitude of the correction for acceleration compensation to be maximum at the highest frequency of forced oscillations. These factors should increase the RMS value of the output from the force transducer even further. It is therefore significant to note a broad peak in the spectral density function for Point E (refer to Fig. 1) which is located at the mean position of the end of the separation zone $(5,6)$. This peak lies in the range $0.02 < fb/U_0 < 0.06$. Since previous work $(5)$ has determined that the effects of the oscillation of the instantaneous stagnation points are transmitted upstream, it is reasonable to assume that vibration of the gate model at frequencies in the range of the broad peak would reinforce the mechanism of the fluid-dynamic feedback loop, resulting in a slightly increased RMS value of the fluctuating force. Once the frequency of the forced oscillation is outside the critical range, no reinforcement should occur; therefore a slight drop in RMS value is observed.

**Summary.** The oscillatory tests for a ratio of gate projection to bottom width $d/b = 1$ demonstrate that: 1) The free shear is highly sensitive to oscillation of the leading edge of the gate model; 2) Large amplitude fluctuations of almost sinusoidal character
are induced by the vortices generated at the frequency of the forced oscillation; and 3) If the dimensionless forcing frequency lies in the range $0.03 < fb/U_o < 0.06$, a slight reinforcement of a fluid-dynamic feedback loop is suggested.

**Gate Model Stationary: $d/b = 3$**

Experimental results obtained for a geometric ratio of $d/b = 1$ have shown that the forces induced on the bottom surface of the gate model become independent of the lateral dimension, $L$, of the area, $A$, on which the forces are measured for values of $L$ near six inches. It was decided to perform experiments for a geometrical ratio $d/b = 3$ with only one plate, six inches in length since the results so obtained should be generally representative of the flow-induced forces. Measurements acquired with a pressure transducer located on the bottom face of the gate (Point B, Fig. 1) showed the relative RMS value of the pressure fluctuations for $d/b = 3$ to be twice the value measured for $d/b = 1$. A similar increase in the RMS value of the fluctuating force was expected, and as a comparison of Fig. 22 and Fig. 15 shows, the RMS value of the fluctuating force is about 2.4 times the value measured for $d/b = 1$. The significance of this fact was made painfully evident in the course of the experiments. Shortly after tests with $d/b = 3$ began, the joint in the connector between the plate and the force transducer failed (Fig. 6) as a consequence of the increased hydrodynamic loading. The resulting delay in the experimental program was a small scale experience in the undesirable consequences of flow-induced loading!
The observed increase in RMS values of both the pressure fluctuations (5,6) and force fluctuations (Fig. 15 and Fig. 22) is a result of the unstable reattachment phenomenon, or equivalently, the establishment of a fluid-dynamic feedback loop. Reattachment and subsequent oscillation of the fluid near the face of the rigid gate model for \( d/b = 3 \) not only increases the intensity of the measured force fluctuations, but also changes the composition of the frequency spectrum compared with results for \( d/b = 1 \) (Figs. 19 and 24).

The fact that the relative RMS value of the pressure fluctuation increases by a factor of 2 with a change in gate geometry from \( d/b = 1 \) to \( d/b = 3 \) and that the corresponding relative RMS value of the fluctuating force increases by a factor of 2.4 appears contradictory in view of previous discussion of the effects of the size of the transducer face on the RMS values measured. It may be recalled that the conclusions drawn from the discussion mentioned above were based on the assumption that the pressure fluctuations were stationary in space. Certain statistical parameters (e.g., the RMS value of the pressure fluctuation, or the spectral density function) were assumed to be independent of spatial position. A brief consideration of the physical aspects of the flow over the gate model indeed shows that spatial nonstationarity is an important factor.

Let us consider the pressure fluctuations sensed by a small pressure transducer as it is moved from place to place on the bottom face of the gate model with \( d/b = 3 \). The mean flow pattern is nearly two-dimensional. Because the statistical parameters used to describ
the pressure fluctuations are temporal averages, as the pressure transducer is moved in the lateral direction only, the same value of the various parameters will be recorded. That is, the pressure fluctuations in the lateral direction are spatially stationary.

If the transducer is moved in the longitudinal, or x direction only, the situation is much more complex. Near the leading edge of the gate, the high-frequency pressure fluctuations are almost completely absent since the breakdown of the eddies into the fine-scale structure responsible for the high-frequency oscillation occurs, relatively speaking, quite far away from the pressure transducer. Furthermore, the amplitude of the fluctuation near the leading edge will be small because the eddy structure is not fully developed. This means that the RMS value of the pressure fluctuation will likewise be small. When the pressure transducer is located near the trailing edge of the gate model, the interaction of the free shear layer with the boundary is the predominant contributor to the pressure fluctuations. It is clear that the intensity of the pressure fluctuations produced by the attachment and reattachment of the free shear layer to the gate model results in a much larger RMS value of fluctuating pressure and a significantly different spectral density function than would be obtained at the leading edge of the gate. The pressure fluctuations are therefore nonstationary in the longitudinal direction.

If it were assumed that the measurements obtained from the pressure cell on the face of the gate model at Point B are truly
representative of the pressure fluctuations over the entire face of the gate, and that the pressure fluctuations are nearly spatially stationary, then one might conclude that the factor of two observed as \( d/b \) was changed from 1 to 3 would also be an upper bound for similar measurements of the flow-induced forces. In fact, such measurements yielded a factor of 2.4. Fortunately, the correct order of magnitude of the increase in RMS value of the force fluctuations with a change in geometry from \( d/b = 1 \) to \( d/b = 3 \) was indicated by the pressure transducer. However, it is quite clear that if there is significant spatial nonstationarity, as is generally the case in shear flows related to a practical hydraulic structure, one must be careful in drawing conclusions from point measurements.

A judicious extrapolation of the spectral density function of the pressure fluctuations to obtain an estimate of the characteristics of the force fluctuations becomes even more tenuous in the face of the spatial nonstationarity of the pressure fluctuation. As reported elsewhere (5,6), a dominant frequency in the pressure fluctuations was observed on the face of the model gate for \( d/b = 3 \). This dominant frequency is a result of the selective amplification mechanism present in the feedback loop, a concept discussed in Chapter I. Because the pressure fluctuations over the small face of the pressure transducer are highly correlated, one would expect that relative magnitude of the peak in the spectral density function would decrease with increasing size of the transducer face. That is, although there is a dominant frequency in the pressure fluctuations at every point on
the face of the gate model for \(d/b = 3\), the phase relationship of
the dominant component is a random function of space and time, and
for spatially stationary pressure fluctuations, the relative
intensity of the dominant component should decrease with increasing
size of the transducer face.

Again, the experimental data show that the effects of significant
nonstationarity space are important, for, as Figs. 23 and 24 reveal,
the relative intensity of the dominant frequency components is
essentially the same for both the pressure cell and the force trans-
ducer. The fact that the relative intensity of the dominant components
of the force fluctuation is, if anything, slightly higher than compo-
nents of the pressure fluctuations may be attributed to two considera-
tions. First, the correlation of the dominant components in the
lateral direction is probably higher than the overall correlation of
the force fluctuation indicated by the data for \(d/b = 1\), and second,
the intensity of the dominant force fluctuation over the downstream
one-third of the face of the gate model is relatively larger than at
Point B where the pressure measurements were obtained. However, to
offset the anticipated trends in the data, these fluctuations must
be very substantial indeed.

Although the results discussed above do not completely define
the role of the spatial nonstationarity of the pressure fluctuations,
it is clear that the measurements presented here are representative
of the flow-induced forces on a gate model. The results obtained
from the pressure transducer, while useful in general, must be interpreted with great care.

Cavitation Flow: \( d/b = 3 \). As anticipated from the data obtained for \( d/b = 1 \), the variation of the relative RMS value of the fluctuating force with cavitation number, shown in Fig. 22, showed the same general trends as the variation of the relative RMS value of the pressure fluctuations with cavitation number obtained with a pressure transducer located on the bottom face of the gate model at Point B (6). Both sets of data show a definite maximum RMS value at a cavitation number, \( K \), of approximately 2.22. A significant change in the characteristics of the fluctuating force as well as the fluctuating force as well as the fluctuating pressure (6) was observed at \( K = 2.18 \). For both cases, severe bursts of sound with a definite periodicity could be heard. These periodic bursts were easily observed on oscilloscope traces of the voltage representing the force fluctuations depicted in Fig. 25a through Fig. 25d. Note that Fig. 25d, which shows the bursts more clearly, is the same as Fig. 25c, except for a scale change which presents a clearer picture of the frequency of the oscillations. Also compare the oscillograms of the non-cavitating flow-induced forces with the oscillogram depicting cavitating conditions.

Because of the distinct differences which were observed, spectral density functions of the fluctuating force were obtained for several cavitation numbers in an attempt to clarify the role of cavitation with respect to the flow-induced forces. A comparison of the spectral
density function of the force fluctuations for $k = 2.52$ (Fig. 26) with the spectrum obtained with no cavitation (Fig. 24) shows that the increase in the RMS value of the fluctuating force is a result of an increase in the intensity of high-frequency fluctuations and is not a consequence of a reinforcement of the fluid-dynamic feedback loop. The high-frequency range of the spectral density function has not been plotted because the difference in the value of nondimensional spectral density for non-cavitating conditions (0.06) compared to the value for the cavitating condition (0.5) would hardly be discernable on the scales chosen for Figs. 24 and 26. Furthermore, the peak in the spectral density function of the force fluctuations for $k = 2.52$ is less than the peak in the spectral density function for non-cavitating flow. In fact, cavitation seems to interfere with the reattachment phenomenon, actually decreasing the relative magnitude of the dominant oscillation—a statement which is contrary to the conclusions noted for $d/b = 1$. Physical reasoning shows that cavitation plays a different role in the two cases.

A continuous decrease in cavitation number eventually results in supercavitating flow and the establishment of a stable vapor pocket behind the normal wall for both geometries. For $d/b = 1$, cavitation helps to organize the dominant eddy structure within a free shear layer which has not been influenced by the gate thickness, $d$, (Fig. 4a) and stabilizes the free shear layer against the natural tendency toward breakdown. Cavitation thus acts within a flow field in which no other mechanisms are actively competing for control.
The reattachment phenomenon present for \( d/b = 3 \), on the other hand, is not compatible with the general trend toward establishment of a vapor cavity, for with supercavitation, the free shear layer becomes detached from the gate model, forms a surface of separation, and completely eliminates the phenomenon of unstable reattachment. The important point is that for \( d/b = 3 \) the later stages of cavitation can only tend to interfere with the fluid-dynamic feedback and inhibit the dominant oscillation, whereas for \( d/b = 1 \), cavitation itself is the organizing mechanism.

During the stage of incipient cavitation, when the cores of the dominant eddies are the first to cavitate, it is possible that the feedback mechanism is augmented slightly. However, the presence of the vapor-water mixture cushioning the forces induced on the bottom surface of the gate completely overwhelms any effects of an increased organization of the eddies in the free shear layer. It is an inescapable conclusion, based both on physical reasoning and the experimental evidence displayed in the spectral density function of the force fluctuations, that for \( d/b = 3 \), cavitation in itself does not enhance the fluid-dynamic feedback, but on the contrary, diminishes the relative intensity of the dominant flow oscillations. Furthermore, the increase in RMS value of the force fluctuations with a decrease in cavitation number for \( d/b = 3 \) may be attributed to the high-frequency fluctuations associated with the phenomenon of cavitation. One further observation should be noted. The increase in the relative RMS value of the fluctuating force from non-cavitating
conditions to the $K = 2.52$ for $d/b = 1$ is 30%, whereas for $d/b = 3$
the corresponding increase is only 6.2%. It may be concluded
from this and the spectral density functions (Figs. 24 and 26) that
part of the increase in RMS value for $d/b = 1$ is attributable to
the high-frequency contribution to the spectrum (Fig. 25b) associated
with the acoustic aspects of cavitation, and part is attributable to
an increased organization of the dominant eddy structure of the shear
flow.

A previous paper (6) indicated that a secondary peak at $K = 2.18$
was observed in the variation with cavitation number of the relative
RMS value of the pressure fluctuations as obtained with a pressure
cell on the bottom face of the gate model (Point B) for $d/b = 3$.
A secondary peak did not appear in number data of the fluctuating
force on the 6-inch long plate, probably because of the variation
of the cavitation number from one end of the plate to the other. The
peak is discernible only over a very small range of cavitation
numbers, and as a consequence of gravitational effects resulting
from the position of the normal wall in the test section, this peak
becomes smoothed and indistinguishable from the primary maximum
near $K = 2.22$. Because of the periodic bursts of noise audible at
$K = 2.18$, a spectrum of the fluctuating force was obtained and is
depicted in Fig. 27. This spectrum is markedly different from any
of the other spectra. For a cavitation number of 2.18, there is more
vapor than fluid present near the force-sensitive plate. The
presence of the vapor-fluid mixture has significantly cushioned the
force fluctuations so that only a vestigal remnant of the dominant peak at $fb/U_o = 0.017$ appears. The large peak in the spectrum at $fb/U_o = 0.190$ was quite unexpected. A careful investigation of the water-tunnel motor, speed reducer, and fluid drive utilizing an accelerometer and spectral analysis as well as magnetic timing devices failed to reveal any source of vibrations at the frequency indicated by the peak in the spectrum. The natural frequency of the system which is composed of the 29" x 24" backplate and forms the entire east side of the water-tunnel test section was determined by striking the backplate and observing the force fluctuations as displayed on an oscilloscope. These traces, an example of which is shown in Fig. 28, indicated that the natural frequency of the backplate mounted in the water tunnel was almost coincident with the frequency indicated by the peak in the spectral density function at $fb/U_o = 0.190$. The occurrence of this unexpected peak is therefore the result of a characteristic of the system and not of cavitation in general. However, one may conclude that for certain values of the cavitation number, cavitation can cause very periodic fluctuations at some natural frequency of the system or components thereof which in turn could lead to very serious flow-induced vibrations. (Probably in this particular instance, the self-control was predominantly a combination of fluid-resonant and fluid-elastic feedback.)

**Forced Oscillation of the Gate Model: $d/b = 3$**

Because of the practical implications of the results obtained for a ratio of wall thickness to projection into the flow for
d/b = 3 with fixed, rigid boundaries, an extensive series of tests was performed to determine the effects of gate displacement on the flow-induced forces and to gain some insight into possible fluid-elastic effects. Data were recorded for forced oscillations with amplitude ratios of $a_o/b = 0.01, 0.02, \text{ and } 0.03$, and frequencies ranging from 1.5Hz to 15Hz. The acceleration compensation procedure proved unsatisfactory for $a_o/b = 0.03$ at all frequencies, and for $a_o/b = 0.02$ and frequencies greater than 8Hz. The data for $a_o/b = 0.02$ and frequencies less than 8Hz are not of high quality, but are sufficiently reliable to warrant inclusion here.

As discussed previously, there are some difficulties in determining precisely how to compensate for effects of virtual mass. There is no question that the use of a voltage proportional to the displacement to compensate for inertial effects is valid; however, the virtual mass aspects remain uncertain. For $d/b = 1$, the magnitude of the necessary compensating voltage was small in comparison with the magnitude of the voltage representing the force fluctuations. Furthermore, the RMS value of the uncompensated voltage was always greater than the RMS value of the compensated voltage.

Although the mass of the 6-inch long plate for $d/b = 3$ was approximately 3 times the mass of the 6-inch long plate for $d/b = 1$, the increase in RMS value of the flow-induced forces again made the required compensating voltage small in comparison to that corresponding to the flow-induced forces. There was no discernible difference between the RMS values of the uncompensated and compensated voltage.
for \( a_o/b = 0.01 \). Any statistical measurement varies from trial to trial, even with the 136-second averaging period used in this investigation. The scatter from measurements of the uncompensated RMS value was within the same range as the scatter from the compensated voltage for \( a_o/b = 0.01 \). However, for greater amplitude ratios, especially at higher forcing frequencies, the correction for virtual mass became appreciable, and the RMS value of the compensated voltage did become greater than the uncompensated value. This observation led to the conclusion that a phase shift between the compensating signal and the virtual mass effect had occurred. As a consequence, the compensating voltage effectively added to the signal voltage. Although some of the data are not suitable for presentation, even these unacceptable results qualitatively substantiate all of the important conclusions drawn from the data for \( a_o/b = 0.01 \).

No functional relationship appears to exist between the relative RMS value of the fluctuating force and the dimensionless forcing frequency for an amplitude ratio of 0.01 (Fig. 29). This result is in contrast to data obtained from circular cylinders, where, as the frequency of the forced oscillations approaches the dominant frequency associated with the wake behind the cylinder, a significant increase in the RMS value of the included force is observed (20,21).

There is a fundamental difference in boundary geometry which accounts for this difference in behavior. The origin of the free shear layer is fixed at the gate's leading edge, whereas the origin of
the free shear layer is not so constrained on the surface of the circular cylinder. As a circular cylinder oscillates, the induced pressure field organizes the motion of the origin of the free shear layer causing a marked increase in spanwise correlation. The increased coherency along the cylinder is transmitted to the wake, and permits more efficient transfer of energy to the dominant eddy structure in the wake of the cylinder. Thus, the phase relationship between the induced forces in the lateral direction is enhanced, resulting in greater RMS value of the force on the cylinder. It is not surprising that the peak RMS value of the induced force which occurs at the "dominant" frequency of the wake increases with increasing amplitude of the forced oscillations, since larger amplitudes exercise more control over the motion at the origin of the free shear layer.

Because the origin of the free shear layer is fixed at the leading edge of the gate model, forced oscillations can do nothing to increase the lateral correlation at the leading edge of the gate. Furthermore, self-control of the free shear flow involving fluid-dynamic feedback in the form of unstable reattachment of the free-shear layer to the gate model is present. One may surmise that the flow has established the optimum process for accomplishing the control, and that any attempt to increase further the efficiency of the feedback loop would require an additional input of energy at precisely the dominant frequency associated with the flow. Oscillation of the gate model at frequencies other than the "dominant"
frequency conceivably could lead to interference and a decrease in the RMS values of the flow-induced forces.

Indeed, observations with a vibrating circular cylinder substantiate such reasoning (21). Forced vibration at the dominant frequency results in a significant increase in the induced forces, but vibration at frequencies near the dominant frequency results in a RMS value of induced loading which is less than the value measured without forced vibration. The RMS value of the induced force approaches the observed stationary values as the forcing frequency is moved further from the dominant frequency.

If such interference is present for the vibrating gate model, its effect is hardly discernible. The relative RMS values of the fluctuating force for the lowest frequencies of forced oscillation are slightly less than the RMS value obtained without oscillation. There is also a very slight rise and fall in the RMS value of the fluctuating force (Fig. 34) near \( f_{ob}/U_0 = 0.017 \) for \( a_0/b = 0.02 \). One of the experimental runs for \( a_0/b = 0.01 \) supported this observation. Because of the resulting scatter of the data for \( a_0/b = 0.01 \) and the uncertainties in compensation for virtual mass effects for \( a_0/b = 0.02 \), one must not read too much into the data. It appears wise to leave the issue open to question.

Spectral analysis of the fluctuating force with forced oscillation of the gate proved to be a necessary tool in determining the characteristics of the induced loading, particularly in view of the absence of a strong relationship between the RMS value of the
fluctuating force and the forcing frequency. Spectra were obtained at nondimensional forcing frequencies of 0.00776, 0.01695, 0.0404, and 0.614, corresponding to physical frequencies of 1.92, 4.16, 10, and 15 Hz respectively, with an amplitude ratio $a_o/b$ of 0.01. These results are depicted in Figs. 30-33.

It is interesting to note that there is no evidence of a peak in the spectral density function at the nondimensional forcing frequency of 0.00776 (Fig. 30), as compared to the obvious peaks at forcing frequencies of 0.0404 and 0.0614, Figs. 32 and 33. The accelerations imparted to the fluid by the oscillating gate model are proportional to the square of the frequency of the forced vibration. Therefore, for the small amplitudes ($\pm 0.01$ inch) and low frequency (1.92 Hz) it is not surprising that this small periodic signal becomes lost in the low-frequency random fluctuation already present. Although spectral analysis is an essential tool in the interpretation of random phenomena, it is no panacea, and definitely has limitations. Other techniques would be required to determine the presence of a periodic waveform obscured by "noise." Of course, once the combination of amplitude and forcing frequency produces an acceleration of sufficient magnitude, a peak appears in the spectral density function, as shown on Figs. 32 and 33, for forcing frequencies of 10 Hz and 15 Hz respectively. Note the difference in magnitude of the spectral density at the different forcing frequencies.

By far the single most important conclusion in this entire investigation can be drawn from Figs. 24 and 30 through 33 inclusive.
It is evident that oscillation of the gate model at nondimensional forcing frequencies other than the dominant frequency at $f_b/U_o = 0.017$ does not significantly disturb the self-control involving fluid-dynamic feedback. The magnitude of the dimensionless spectral density function at the dominant frequency with forced oscillation is nearly the same as the value without forced oscillation; only a small decrease is noticeable for $f_o = 15$ Hz. There is no change for $f_o = 10$ Hz. Therefore, for $d/b = 3$, not only does a flow instability exist which may excite flow-induced oscillations, but the mechanism of this instability does not break down once the structure begins to vibrate. The dominant frequency component remains present in the fluctuating forces induced on the bottom face of the gate model to serve as a forcing function and maintain or further augment the structural vibration.

When the forcing frequency, $f_o$, coincides with the dominant frequency, the spectral density function of the fluctuating force indicates that the fluid-dynamic feedback loop has been reinforced. The peak value of the spectral density for a forcing frequency of 10 Hz is about six nondimensional units larger than the force fluctuation present without forced oscillation. Assuming that there are no effects resulting from an interaction between the flow and the gate model, the forces induced by oscillation of the gate model would be proportional to the square of the forcing frequency. For the lower forcing frequencies, the addition to the spectral density function would necessarily be smaller than for the higher forcing frequencies.
Therefore, the increase in spectral density for a nondimensional forcing frequency, \( fb/U_0 = 0.017 \) of about 12 nondimensional units, indicates that some reinforcement of the feedback loop occurred. The reinforcement is not as spectacular as has been observed during the forced vibration of circular cylinders, but is apparent nonetheless. Thus, if the natural frequency of the structure coincides with the dominant frequency observed in the spectrum for \( d/b = 3 \), an increase in the flow-induced loading at the dominant frequency may be expected. Because of the stochastic nature of the exciting force, the amplitude of the resulting vibrations would not be as large as predicted by classical, deterministic analysis of the problem. This fact is of little consolation, however, since the quasi-steady amplitude of the flow-induced vibration would probably be of sufficient magnitude to cause failure either by direct overstress or fatigue.

The explanation for the strong reinforcement observed with the vibrating circular cylinder lies in the increased spanwise correlation caused by the forced oscillations, as discussed previously. The origin of the free shear layer is fixed at the leading edge of the gate model. No increase in lateral correlation is thus possible at the leading edge. Further, the fluid-dynamic feedback has probably established the most efficient method for the transfer of energy to the fluctuating flow. If this statement is true, one would not expect any increase in the spectral density at the dominant frequency over and above the energy input to the flow via the forced oscillation at
the dominant frequency. A subtle but important point arises. Whereas
the fluid-dynamic feedback loop has most probably established the
optimum control for a ratio of d/b = 3, the geometry corresponding
to d/b = 3 may not establish the optimum feedback loop. That is,
the geometry for which the reattachment of the free-shear layer is
least stable is not d/b = 3, but some other value, such as 2.8, for
example. It is interesting to speculate that for the optimum
geometry, the addition to the spectral density by the forced vibra-
tion at the dominant frequency would not be significantly greater
than that expected from the forced oscillation alone, without any
amplification through the feedback loop. Results obtained with a
pressure transducer located at Point B on the face of the gate model
for d/b = 3 and d/b = 3.5 showed that the nondimensional value of
the dominant frequency remains unchanged with variation in d/b in
this instance. It appears that d/b = 3 is so close to the optimum
gate geometry that any observed differences between the optimum geometry
and d/b = 3 would be minor. The experimental results for d/b = 3
are therefore representative of the critical condition.

A few brief remarks are in order regarding the results for an
amplitude ratio of 0.02 and frequencies less than 8 Hz. The vari-
ation of the relative RMS value of the fluctuating force with
dimensionless forcing frequency, depicted in Fig. 34, shows an
increasing trend with forcing frequency. Because the magnitude of
the compensation for inertia and virtual mass is greater at the
higher forcing frequencies, the general upward trend is most probably
attributable to the unsatisfactory method of compensation. The
minor variation around $f_b/U_o = 0.017$ has been alluded to previously.
A spectral density function of the fluctuating force with an amplitude
ratio of 0.02 and a nondimensional forcing frequency of 0.00776 is
presented in Fig. 35. It may be recalled that for $a_o/b = 0.01$ no
effect of this forced oscillation was visible in the spectrum.
Doubling the amplitude of a sine wave voltage will increase its
contribution to the spectrum fourfold. Therefore, doubling the
amplitude of the forced oscillation at frequencies other than the
dominant frequency should more or less quadruple the spectral density
at the forcing frequency. The effects of the forced oscillation at
$f_b/U_o = 0.00776$ are now large enough to appear in the spectrum, as
Fig. 35 shows.

**Summary.** For $d/b = 1$ and the gate model stationary, no
tendency toward periodicity was found, in agreement with previous
work (5). A plate length of 6 inches was required for force measure-
ments to be independent of the lateral correlation of the pressure
fluctuations. Cavitation appears to stabilize the dominant eddy
structure against breakdown, as evidenced by the increase in the
low-frequency range of the spectrum. Oscillation of the gate model
with $d/b = 1$ showed that external control produced a periodic force
with a frequency coincident with the frequency of the forced oscil-
lations.

The existence of self-control involving fluid-dynamic feed-
back was again confirmed through measurement of the fluctuating force
for \( d/b = 3 \). Because of the spatial nonstationarity of the pressure fluctuations, the force measurements obtained in the present study give a better measure of the effects of flow-induced forces than do point measurements obtained with a pressure transducer. Forced oscillation of the gate model for \( d/b = 3 \) demonstrated that displacement of the gate model at frequencies other than the dominant frequency shown in the spectrum for \( d/b = 3 \) and the gate stationary did not interfere with the fluid-dynamic feedback loop, whereas oscillation at the dominant frequency did indicate some reinforcement.

The preceding sections have presented detailed explanations and discussions of a variety of phenomena associated with the forces induced by flow over a model gate. The following chapter endeavors to consolidate these results which are of more practical significance.
CHAPTER VI. APPLICATION OF PROBABILISTIC ANALYSIS TO THE VIBRATIONAL RESPONSE OF A SIMPLIFIED SYSTEM

The purpose of this section is threefold: first, to summarize those conclusions presented in previous chapters which are of particular significance from a practical viewpoint; second, to demonstrate the application of the statistical methods used in determining the response of structures to flow-induced forces, with special emphasis placed on the spectral density function; and third, to delineate carefully the limitations of the data obtained in this study with respect to the prediction of structural displacements.

At the outset, it must be understood that the force induced on the gate model is a stochastic (random) process, as are the pressure fluctuations and velocity fluctuations associated with turbulent flow. Qualitatively, a random process may be characterized as follows: suppose that at time, $t_0$, all of the previous values of a force or pressure are known as a function of time. Then at a later time, $t$, it is impossible to predict the exact value of the force or pressure; only an estimate in accordance with the probability distribution of the variable can be made. For example, on the basis of the appropriate statistical analyses, one might state that the probability at a later time, $t$, for the magnitude of the force to be between 50 and 70 lbs is 80%.
The usual engineering problems involve analysis by deterministic rather than probabilistic methods. The simplest illustrative example which may be found in almost every textbook on the subject of random vibrations is the spring, mass, dashpot system, shown schematically in the sketch below. The spring is assumed to be linear, with spring constant k, the damping is viscous, with damping coefficient c, and the mass, m, is acted upon by a force which varies with time, f(t). The differential equation for the motion of this system is

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = f(t).$$

which is usually rewritten as

$$\frac{d^2x}{dt^2} + 2 \zeta \omega_n \frac{dx}{dt} + \omega_n^2 x = \frac{f(t)}{m}.$$

Herein \(\omega_n = \sqrt{k/m}\) is the undamped natural frequency of the system and \(\zeta\), the damping ratio, is the ratio of the actual damping to the critical damping. The solution to this second order, linear differential equation for a sinusoidal forcing function, f(t) = \(F_0 \exp(i\omega t)\) is

$$x = A \exp[-(\zeta + i\sqrt{1-\zeta^2})\omega_n t] + \frac{F_0 \exp(i\omega t)}{m(\omega_n^2 - \omega^2 + 2i\zeta\omega_n \omega)} [6]$$

Here A is a complex constant which is determined from the initial conditions. This solution has been separated into two parts. The first term on the right represents the transient response of the system to the forced vibrations, and the second represents the so-called steady-state response. The denominator of the second term, designated by \(Z(\omega)\),
\[ Z(\omega) = m\left(\omega_n^2 - \omega^2 + 2i\delta \omega_n \omega\right) \]

is called the mechanical impedance of the system, and its reciprocal,
\[ H(\omega) = \frac{1}{Z(\omega)} = \frac{1}{m\left(\omega_n^2 - \omega^2 + 2i\delta \omega_n \omega\right)} \]

is called the frequency response function or the transfer function of the system.

The method outlined above is the classical deterministic approach to vibration problems. That is, if the forcing function, \( f(t) \), is a known function of time, then for a given spring constant, damping coefficient, and initial conditions, the displacement of the mass from its equilibrium position can be determined as a unique function of time by eq. [6]. For a deterministic forcing function, the response of this simple system is also deterministic.

As a corollary to the last statement, a stochastic forcing function produces a stochastic response. The differential equation of motion becomes a second order stochastic differential equation
\[ \frac{d^2 X}{dt^2} + 2\delta \omega_n \frac{dX}{dt} + \omega_n^2 X = \frac{F(t)}{m} \]

where \( X \) is the random process representing the displacement of the mass from its equilibrium position, and \( F(t) \) is the random process representing the force acting on the mass. The average engineer is now faced with the uncomfortable situation where the forcing function can be characterized only by statistical parameters. The problem is, first, to obtain some meaningful parameters describing the random input to the system, and second, to find some method to determine the
characteristics of the displacement, or output of the system from
the known input characteristics.

To simplify the discussion, the random process, \( X(t) \), is assumed
to be stationary, ergodic, and have a zero mean value. The condition
of a zero mean value is self-explanatory. Stationarity is a property
which guarantees that the values of the statistical parameters are
independent of the time at which the measurement of the average is
begun. Ergodicity means that the value of the statistical parameters
may be obtained by temporal averages. A more detailed discussion of
these properties will be found in Appendix A, and a more formal
presentation is available elsewhere (23, 24). Because the intent
here is not to present a condensation of the theory of random
processes, some of the following statements will be given without
proof. A random process may be described by moment functions of
various orders which are in themselves functions of the various
probability density functions specifying the random process. The
first and second order moment functions are the most practically
significant. For a stationary, ergodic process, the first order
moment is the mean, defined as

\[
\overline{X(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t) \, dt
\]

and the second order moment is the auto-covariance function, defined
as

\[
C(\tau) = \overline{X(t)X(t+\tau)}
\]
The bar over the quantity denotes a temporal average. The value of the auto-covariance for $\tau = 0$ is the variance of the process:

$$\sigma^2 = C(0) = \langle X(t)X(t) \rangle = \langle X(t)^2 \rangle$$

The square root of the variance is the RMS value of the process.

For an ergodic process with zero mean, the autocorrelation function is related to the spectral density function by the Fourier transformation:

$$P(f) = \int_{-\infty}^{\infty} C(\tau) e^{-i2\pi f\tau} d\tau$$

$$C(\tau) = \int_{-\infty}^{\infty} P(f) e^{i2\pi f\tau} df$$

Therefore, the auto-covariance and the spectral density function contain the same information, but display it in different forms. It may be shown (refer to Appendix A) that the physical spectral density function used in this investigation may be expressed using the variance as a normalizing parameter as:

$$R(\tau) = \frac{C(\tau)}{C(0)} = \int_{-\infty}^{\infty} \Phi(f) \cos(2\pi f\tau) df$$

$$\Phi(f) = \frac{P(f)}{C(0)} = \frac{4}{\pi} \int_{0}^{\infty} R(\tau) \cos(2\pi f\tau) d\tau$$

The first and second order statistics of a random process are important because: 1) For some random process the statistical properties of higher orders can be computed if the first and second order statistical properties are known (the Gaussian process, for example); 2) A meaningful physical interpretation is easily attached to these functions; and 3) The computations of higher order statistical properties from recorded data exhibit a great deal of scatter. Therefore, the first and second order statistics are the
most meaningful and are also the most readily obtained.

For an ergodic process with a zero mean value, the autocorrelation function, or equivalently, the spectral density function, is the most important description of the random process. The variance or mean square value of the process is a measure of the deviation of the process from its mean value. A large variance indicates rather wide deviations from the mean. A large RMS value (square root of the variance) therefore indicates a high intensity in the fluctuations of the process. Or in more concrete terms, a large RMS value of the fluctuating force means that the fluctuations are intense, and that significant deviations from the mean are quite frequent.

An interpretation of the spectral density function is not an elementary problem. Part of the difficulty lies in the fact that classical harmonic analysis does not apply to a stationary random process; that is, one cannot write a Fourier series expansion for a random process, unless one is willing to permit the coefficients of the series to be random processes. Perhaps the most meaningful interpretation can be obtained by setting \( \tau = 0 \) in (7) above,

\[
\frac{C(0)}{C(0)} = R(0) = 1 = \int_{-\infty}^{\infty} \Phi(f)df = \int_{0}^{\infty} \frac{P(f)}{C(0)} df
\]

Recalling that \( C(0) \) is the mean square of the random process, then \( P(f)df/C(0) \) represents the ratio intensity of the fluctuations within the bandwidth, \( df \), to the total intensity of the fluctuations. The spectral density function, \( \Phi(f) \), is a statistical decomposition
of the random process with respect to frequency. The ordinate of the function $\Phi(f)$ is a measure of the relative intensity of the fluctuations at the frequency, $f$.

All phase information is lost in spectral analysis; thus, the spectrum fails to differentiate between sine and cosine waves. It may be shown that a pure sine or cosine wave theoretically shows up in the spectrum as an infinite spike, known as a Dirac delta function. Therefore, not only does the spectrum display the distribution of the relative intensities with respect to frequency, but indicates the presence of periodic or nearly periodic components as well. Such components appear as high narrow peaks in the measured spectra.

With meaningful parameters at hand to characterize a random process, let us turn to the problem of determining the characteristics of the random motion of the mass, $m$, in the system discussed previously. Because a linear system with one degree of freedom is well documented (24,25), it is sufficient to merely state the necessary conclusions. The mean square (variance) of the displacement may be calculated from the spectral density of the forcing function, $F(t)$, and the frequency response function by the relationship

$$X(t)^2 = \int_{-\infty}^{\infty} |H(f)|^2 P(f) df$$  \[8\]

where $|H(f)|^2$ is the product of the transfer function, $H(f)$, and its complex conjugate. The spectral density function of the response is related to the spectral density function of the forcing function by (24,25)
\[ P_x(f) = |H(f)|^2 P_f(f) \]

Herein the subscript, \( F \), refers to the forcing function and the subscript, \( x \), refers to the displacement of the mass, \( m \). The function, \( |H(f)|^2 \), is called the transmittancy function or system function and permits only certain frequencies of the input spectrum, \( P_f(f) \), to appear in the output spectrum, \( P_x(f) \) - a filter in the parlance of electrical engineering. The transmittancy function for the linear system is:

\[
|H(f)|^2 = \frac{1}{m^2 \left( \frac{2\pi f_n}{2\pi f_n} \right)^4 \left[ \left( 1 - \left( \frac{2\pi f}{2\pi f_n} \right)^2 \right)^2 + 4 \left( \frac{2\pi f}{2\pi f_n} \right)^2 \right]} 
\]

Normalizing the function with the value, \( |H(0)|^2 = \frac{m^2 \left( \frac{2\pi f_n}{2\pi f_n} \right)^4}{k^2} \), the following expression results:

\[
\frac{|H(f)|^2}{|H(0)|^2} = \frac{1}{\left[ \left( 1 - \left( \frac{2\pi f}{2\pi f_n} \right)^2 \right)^2 + 4 \left( \frac{2\pi f}{2\pi f_n} \right)^2 \right]} 
\] \[ \tag{9} \]

Equation [9] has been plotted on Fig. 36 for damping ratios of 0.05, 0.10, and 0.15. Notice that the peak in the function at \( f = f_n \) reduces significantly with an increase in damping ratio.

The expression [8] for the mean square value of the displacement may be made nondimensional by dividing both sides of the equation by \( |H(0)|^2 \) \( k \) and the mean square value of the fluctuating force, \( \overline{F^2} \)

\[
\frac{\overline{X^2}}{k \overline{F^2}} = \int_0^\infty \frac{|H(f)|^2}{|H(0)|^2} \frac{U_b}{U_b} \overline{F(b)} \overline{f(b)} \, d\left( \frac{fb}{U_b} \right) 
\] \[ \tag{10} \]

The algebraic manipulations may be found in Appendix C.
A typical spectrum of the fluctuating force for $d/b = 3$ and $l/b = 6$ with the gate model stationary was chosen (Fig. 24) and a numerical integration of the expression for $X'^2$, [10], was performed. It was assumed that the upper limit of $fb/U_o = 1.014$ or $f = 250$ Hz. was sufficient because the value of the spectral density at frequencies greater than 100 Hz. is of the order of .05. Furthermore, the transmittancy function decreases in inverse proportion to $f^4$ at frequencies moderately removed from $f_n$. Since the value of $f_n$ used in this study was less than 12 Hz., an upper limit of 250 Hz. is reasonably accurate as a limit for the integration.

The relative mean square displacement given by eq. [10] has been plotted as a function of the ratio of the natural frequency, $f_n$, of the linear spring, mass, dashpot system to the dominant frequency, $f_d$, in the spectrum of the force fluctuations for damping ratios of 0.05, 0.1, and 0.15 (Fig. 37). For a lightly damped system, $\zeta = 0.05$, a very significant increase in the relative mean square displacement occurs for $f_n/f_d = 1$, as would be expected from the characteristics of the spectrum and the transmittancy function. The upward trend of the three plots in Fig. 37 for a ratio of $f_n/f_d$ less than about 0.25, particularly evident with $\zeta = 0.05$, is a result of the relatively large values of the spectral density function for $fb/U_o$ less than 0.004, as shown in Fig. 24. Because of the uncertainties associated with estimates of the spectral density function near zero frequency (35), this upward trend is probably spurious.
Mean square values do not present a complete picture of the response, however. The spectrum of random displacement may be found from the relationship

\[
\frac{U_0}{b} \Phi \left( \frac{fb}{U_0} \right) = \frac{U_0}{b} \Phi \left( \frac{fb}{U_0} \right) \frac{|H(f)|^2}{|H(0)|^2} \left[ \frac{k^2 F_{r}^2}{X_{r}^2} \right]
\]

[11]

The details of the derivation of this expression are presented in Appendix C. Spectra of the random displacement of the mass for damping ratios of 0.05, 0.10, and 0.15 and natural frequencies corresponding to \( f_n/f_d \) of 0.4, 1, and 2.4 are depicted on Figs. 38-40. These spectra were computed with the same typical spectrum of the fluctuating force used in evaluating eq. [10].

For a natural frequency less than the dominant frequency the spectra for the three damping ratios indicate an almost periodic excitation, as evidenced by the large peak in the spectrum. Again it must be emphasized that the spectra plotted in nondimensional form indicate the relative intensity of the fluctuations at a given frequency, \( f \). For each value of \( \zeta \), the displacement tends to be more periodic for \( f_n < f_d \) than for either of the other two spectra shown as is demonstrated by the greater peakedness for the respective values of \( f_n \). Although the displacement is almost periodic, the mean square value (Fig. 36) is small in comparison to the values at \( f_n/f_d = 1 \) and 2.4. When the natural frequency, \( f_n \), coincides with the dominant frequency, \( f_d \), a strong tendency toward periodicity, coupled with a large mean square value of the displacement, shows clearly that such a condition must be avoided in design of a structure.
This conclusion was drawn earlier from inspection of the measured spectral density functions of the flow-induced forces. The results in Figs. 37 and 38-40 place this result in better analytical perspective.

For \( f_n > f_d \), the excitation at the dominant frequency, \( f_d \), is attenuated, and a tendency toward vibration at the natural frequency, \( f_n \), is indicated. The logarithmic scale has significantly distorted the area under the curves for \( f_n > f_d \), and it is not at first obvious that there is a much wider range of frequencies present in the displacement spectrum than for either of the other two cases. With a damping ratio of 0.1 and 0.15, the ordinates of the spectral density function of the displacement are of the same order of magnitude for \( f_n > f_d \). The displacement in this case does not show any strong tendency toward periodicity. The comparatively small mean square value would indicate that such displacements could be tolerated.

Admittedly, the analysis just presented is a first order approximation to the response problem. Nevertheless, the information provides a quantitative estimate of the response of a linear system with one degree of freedom. Assuming no cavitation, a knowledge of the oncoming flow velocity, \( U_0 \), and an estimate of the structural properties \( k, c, \) and \( m \), the mean square response of a system with a geometry similar to the geometrical configuration \( d/b = 3 \) can be estimated as follows: first, from Fig. 22, the mean square force acting on the bottom face of the normal wall is obtained. Second, from Fig. 37 and computing \( f_n = 1/2\pi \sqrt{k/m} \) and \( f_d = 0.017 U_0 / b \), the
mean square displacement is calculated. From the damping ratio $\xi$, and possibly with some interpolation, a reasonable picture of the spectrum of the displacement is available from Figs. 38-40. The mean square value (or RMS) and the spectrum should permit a judgment to be made on feasibility of the design.

As far as high-head gates in dams and appurtenant works are concerned, the major U.S. Government agencies working in this field, the U.S. Army Corps of Engineers and the U.S. Bureau of Reclamation, have alleviated their gate-vibration problems by using different shapes of the gate lip (27,28). A standard 45° lip used by the U.S. Army Corps of Engineers, for example, causes separation to occur at the sharp, downstream edge of the gate. The free shear layer has no chance to interact directly with the gate itself, as is the case with a flat-bottomed gate. Any disturbances originating in the free shear layer must be transmitted to the structure through the separation zone behind the gate, which is a relatively weak feedback path, as discussed earlier, and the tendency for the gate to vibrate is much reduced (4). It is clear, both from Corps of Engineers data and the data obtained in this study, that the construction of flat-bottomed, high-head gates is poor design practice and should be avoided. The obvious advantages in the structural framing of the flat-bottom are offset by the undesirable hydraulic characteristics.

Based upon the experimental evidence depicted in Fig. 24 and Figs. 31-33, the vibration problem could be treated as one of forced vibration. If a fluid-elastic feedback loop is established, then
the forcing function will depend on the structural displacement and a much more complex analysis would be required. Similarly, if Coulomb damping were considered, the analysis would be much more complicated. Inclusion of these two factors was beyond the scope of the present study, but is a realizable goal for future work.
CHAPTER VII. CONCLUSIONS

The principal results obtained during the course of this investigation of the flow-induced forces on a square-edged model gate may be summarized as follows:

1. For $d/b = 1$, $b$ = protrusion distance into flow, $d$ = gate thickness, and the gate model stationary, no tendency toward periodicity was found, in agreement with previous work (5). A plate length of 6 inches was required for force measurements to be representative of the flow-induced forces on a wide gate structure.

2. Oscillation of the gate model with $d/b = 1$ showed that external control produces a periodic force with a frequency coincident with the frequency of forced oscillation, conclusively demonstrating that the free shear layer is quite sensitive to disturbance at its origin.

3. The existence of self-control involving fluid-dynamic feedback was confirmed through measurement of the fluctuating force for $d/b = 3$, and a ratio of the length of the force-sensitive plate on the bottom face of the gate, $l$, to gate protrusion, $b$, of six ($l/b = 6$). Spatial nonstationarity of the pressure fluctuations was shown to be an important factor in interpreting data from the pressure transducer. The force measurements obtained in this study
give a better measure of the effects of flow-induced forces than do point measurements obtained with a pressure transducer.

4. Oscillation of the model gate with \( d/b = 3 \) demonstrated that displacement of the wall does not appreciably interfere with the fluid-dynamic feedback loop. The dominant frequency observed for the gate model stationary remains present in the fluctuating force to maintain or further augment vibration.

5. The fact that the spectrum of the fluctuating force remained essentially unchanged during forced vibration, except at the forcing frequency, permits the analysis of the response of the gate model to be treated as a problem in forced vibration. A probabilistic analysis of the response of a linear system with one degree of freedom representing an idealized high-head gate, shows at least to a first order approximation, the characteristics of the response of the system to the fluctuating force. The measurement of the relevant statistical properties of this forcing function was the principal objective of this investigation.

6. The role of cavitation in flow-induced vibration was shown to be strongly dependent on the flow conditions. For \( d/b = 1 \), cavitation definitely organized the structure of those eddies containing most of the turbulence energy. An increase in the fluctuations in the low-frequency range of the spectrum was observed, which substantiates this conclusion. For \( d/b = 3 \), the cavitation interfered with the fluid-dynamic feedback mechanism, resulting in a decrease in relative intensity of the fluctuating force. It was
observed that under certain conditions, cavitation could excite vibration at some natural frequency of the experimental setup.
FIGURE 1. DEFINITION SKETCH
Flow Visualization with the Lustre-Creme Technique from a Film by the Author.

FIGURE 2. DEVELOPMENT OF DISCRETE VORTICES IN A FREE-SHEAR LAYER
With effective control: OSCILLATORY FLOW (Predominantly periodic excitation)

Energy transfer from basic flow to disturbances gives rise to FLUCTUATING FLOW

Fluid-dynamic feedback
Fluid-resonant feedback
Fluid-elastic feedback

Without effective control: TURBULENT FLOW (Aperiodic excitation)

Random disturbance
Periodic disturbance

STEADY SHEAR FLOW

FIGURE 3. SCHEMATIC REPRESENTATION OF THE BASIC MECHANISMS OF CONTROL
(a) NO REATTACHMENT (FOR $T=6b$: $d/b < 2$)

(b) UNSTABLE REATTACHMENT (FOR $T=6b$: $2.5 < d/b < 4.5$)

(c) STABLE REATTACHMENT (FOR $T=6b$: $d/b > 4.5$)

FIGURE 4. EFFECTS OF GATE GEOMETRY ON FLOW REATTACHMENT
FIGURE 5. PHOTOGRAPHS OF THE WATER TUNNEL AND WATER TUNNEL DRIVE SYSTEM.
FIGURE 7. PHOTOGRAPHS OF THE VIBRATORY APPARATUS
FIGURE 8. OSCILLOGRAM SHOWING THE COMPENSATION FOR THE EFFECTS OF ACCELERATION AND VIRTUAL MASS
Figure 10. Spectral density function of the pressure fluctuations with and without cavitation, d/b = 1, Point B (see Fig. 1)
Figure 11. Variation of the relative RMS value of the fluctuating force with \( z/b \) for several cavitation numbers.
FIGURE 12. VARIATION OF THE RELATIVE RMS VALUE OF THE FLUCTUATING FORCE WITH CAVITATION NUMBER, \(d/b = 1, \tau/b = 1\)
FIGURE 13. VARIATION OF THE RELATIVE RMS VALUE OF THE FLUCTUATING FORCE WITH CAVITATION NUMBER, $d/b = 1$, $l/b = 2$
FIGURE 14. VARIATION OF THE RELATIVE RMS VALUE OF THE FLUCTUATING FORCE WITH CAVITATION NUMBER \( \frac{d}{b} = 1 \), \( \frac{L}{b} = 4 \)
FIGURE 16. SPECTRUM OF THE FORCE FLUCTUATIONS, d/b = 1, l/b = 1, NO CAVITATION

SPECTRAL DENSITY FUNCTION: d/b = 1, l/b = 1.

- Run 1
- Run 2
FIGURE 17. SPECTRUM OF THE FORCE FLUCTUATIONS, \( \frac{d}{b} = 1 \), \( \frac{l}{b} = 2 \), NO CAVITATION
SPECTRAL DENSITY
FUNCTION: \( \frac{d}{b} = 1 \)
\( \frac{L}{b} = 4 \)

- Run 1
- Run 2
- Run 3

FIGURE 18. SPECTRUM OF THE FORCE FLUCTUATIONS, \( \frac{d}{b} = 1 \), \( \frac{L}{b} = 4 \), NO CAVITATION
FIGURE 19. SPECTRUM OF THE FORCE FLUCTUATIONS, \( d/b = 1, \ l/b = 6 \), NO CAVITATION
Figure 20. Spectrum of the force fluctuations, \( d/b = 1, \ell/b = 6, a_o/b = 0.01 \), oscillating gate.
Figure 21. Variation of the relative RMS value of the fluctuating force with nondimensional forcing frequency, $d/b = 1$, $l/b = 6$, $a_0/b = 0.01$.
FIGURE 22. VARIATION OF THE RELATIVE RMS VALUE OF THE FLUCTUATING FORCE WITH CAVITATION NUMBER, 
$d/b = 3, l/b = 6$
FIGURE 23. SPECTRUM OF THE PRESSURE FLUCTUATIONS AT POINT B, d/b = 3, AS OBTAINED WITH A PRESSURE CELL AND THE IBM 1801 DATA ACQUISITION SYSTEM
SPECTRAL DENSITY FUNCTION: $d/b = 3, \quad l/b = 6$.

- Run 1
- Run 2
- Run 3

**Figure 24. Spectrum of the force fluctuations with the gate model stationary, $d/b = 3, l/b = 6$**
FIGURE 25. OSCILLOGRAMS OF THE FLUCTUATING FORCE AT SEVERAL CAVITATION NUMBERS, \( d/b = 3, l/b = 6 \)
FIGURE 26. SPECTRUM OF THE FORCE FLUCTUATIONS, d/b = 3, l/b = 6, AND K = 2.52
FIGURE 27. SPECTRUM OF THE FORCE FLUCTUATIONS, $d/b = 3$, $l/b = 6$, AND $K = 2.18$
FIGURE 28. OSCILLOGRAM OF THE BACKPLATE RESPONSE TO IMPULSE EXCITATION
FIGURE 29. VARIATION OF THE RELATIVE RMS VALUE OF THE FLUCTUATING FORCE WITH DIMENSIONLESS FORCING FREQUENCY, $d/b = 3$, $l/b = 6$, AND $a_0/b = 0.01$.
SPECTRAL DENSITY FUNCTION: $d/b = 3$, $a_o/b = 0.01$, $f_o b/U_o = 0.00776$

Run 1
Run 2

$U_o/b$ vs $f_b/U_o$

FIGURE 30. SPECTRUM OF THE FORCE FLUCTUATIONS, $d/b = 3$, $a_o/b = 0.01$, $f_o b/U_o = 0.00776$
Figure 31. Spectrum of the force fluctuations, $d/b = 3$, $a_0/b = 0.01$,
$f_{ch}/U_0 = 0.01695$.
FIGURE 32. SPECTRUM OF THE FORCE FLUCTUATIONS, \( d/b = 3, \) \( a_{c}/b = 0.01, \) \( f_{cb}/U_{o} = 0.0404 \)
FIGURE 33. SPECTRUM OF THE FORCE FLUCTUATIONS, \( d/b = 3, a_o/b = 0.01, \) \( f_o b/U_o = 0.0614 \)
Figure 34. Variation of the relative rms value of the fluctuating force with dimensionless forcing frequency, $d/b = 3$, $l/b = 6$, $a_0/b = 0.02$
FIGURE 35. SPECTRUM OF THE FLUCTUATING FORCE, \( d/b = 3, \ l/b = 6, \ a_o/b = 0.02, \ f_o b/U_o = 0.00776 \)
FIGURE 36. DIMENSIONLESS TRANSMITTANCY FUNCTION FOR THREE DAMPING RATIOS
FIGURE 37. DIMENSIONLESS MEAN SQUARE RESPONSE OF A LINEAR SYSTEM WITH ONE DEGREE OF FREEDOM TO THE FORCE FLUCTUATIONS WHOSE SPECTRUM APPEARS ON FIGURE 24, AS A FUNCTION OF THE DAMPING RATIO AND PROXIMITY TO THE DOMINANT FREQUENCY, $f_d$.
FIGURE 38. SPECTRA OF THE DISPLACEMENTS OF A LINEAR SYSTEM WITH ONE DEGREE OF FREEDOM WITH A DAMPING RATIO OF 0.05 FOR $f_n/f_d = 0.4, 1.0, \text{ AND } 2.4$
FIGURE 39. SPECTRA OF THE DISPLACEMENTS OF A LINEAR SYSTEM WITH ONE DEGREE OF FREEDOM AND A DAMPING RATIO OF 0.1, FOR $f_n/f_d = 0.4, 1.0, \text{ AND } 2.4$
FIGURE 40. SPECTRA OF THE DISPLACEMENTS OF A LINEAR SYSTEM WITH ONE DEGREE OF FREEDOM WITH A DAMPING RATIO OF 0.15, FOR $f_n/f_d = 0.4, 1.0, \text{AND} 2.4$
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APPENDIX A

A DIGITAL TECHNIQUE FOR ESTIMATING

THE SPECTRAL DENSITY FUNCTION
APPENDIX A. A DIGITAL TECHNIQUE FOR ESTIMATING
THE SPECTRAL DENSITY FUNCTION

"Spectral," as noted by Tukey (34), can refer either to spectra (which is the subject of interest herein or to specters, some of which most certainly haunt those who concern themselves with spectral analysis. The purpose of this Appendix is not to exorcise these apparitions, but to show what they are and how one may live more or less comfortably with them. This prospect may be somewhat unsettling to the uninitiated. Since there is no known method for avoiding some of these specters, one must make the best of the present situation. It will be assumed that the reader is not thoroughly familiar with the subject of statistical analysis. Therefore, some elementary concepts and definitions will be presented.

Most practical engineers are accustomed to working with deterministic processes. For example, if a mass suspended from a spring is acted upon by a simple forcing function, say \( F(t) = \sin \omega t \), then the displacement of the mass from its equilibrium position can be uniquely determined as a function of time from the differential equation of motion. This is a deterministic process, since the output of the system can be precisely determined from the input and initial conditions. Although many physical processes may be regarded
as deterministic, a great many others are random. Fluid Mechanics, for instance, is blessed with the well known, but not so well understood random process called turbulence. The velocity or pressure fluctuations associated with turbulent flow are not deterministic. If one knew all previous values of the velocity or pressure at a point in the flow as a function of time, one could not predict precisely the next value, but may only make estimates based on the probability distribution of the variable. In order to characterize a random process, statistical parameters indicating average behavior must be obtained.

Suppose that an experiment involving a random process as an output is performed and that the results, represented by voltages, are recorded. The records, \( X_k(t) \), for several experimental runs might appear as illustrated in Fig. A-1 where the subscript \( k \) denotes a particular experimental outcome or realization. The variable, \( t \), denotes time, since it is convenient to think of electronic signals as functions of time. If the time of observation for \( X_k(t) \) extends from \( -\infty \) to \( +\infty \), \( X_k(t) \) is called a sample function. It should be emphasized that the sample function itself is not the random process. The random process is the collection of all possible sample functions that the experiment might produce. The statistical properties of the random process may be computed by taking averages across the collection (known as the ensemble) of sample functions which make up the random process.
For example, the mean value of the random process at time equal to $t_i$ may be estimated from the relationship

$$\text{ave}\{X_k(t_i)\} = \frac{1}{N} \sum_{k=1}^{N} X_k(t_i)$$ \[1\]

The autocovariance is given by

$$C_{ij} = \text{cov}\{X_k(t_i), X_k(t_j)\} = \text{ave}\{[X_k(t_i) - \text{ave}[X_k(t_i)]][X_k(t_j) - \text{ave}[X_k(t_j)]\}$$ \[2\]

Herein, the brackets $\{\}$ denote the ensemble operations, to be distinguished from temporal operations which are denoted by a bar over the expression.

In general, $\text{ave}\{X(t_i)\} \neq \text{ave}\{X(t_j)\}$. Such a random process is called nonstationary. If the ensemble averages, $\text{ave}\{X(t_i)\}$, $\text{ave}\{X(t_j)\}$, and so on are equal for all $t_i$, and if the value of
the autocovariance, \( \text{cov} \{ X(t_i), X(t_j) \} \) does not vary with \( t_i \), then
the process is said to be weakly stationary or stationary in the
wide sense. Other higher order statistical parameters may be con-
structed, and if the value of these parameters are time invariant,
the process is said to be strongly stationary, or stationary in the
strict sense. For practical purposes, if a random process is weakly
stationary, it is reasonable to assume that it is strongly stationary
as well. For both weakly stationary and strongly stationary processes,
the mean value \( \text{ave} \{ X(t_i) \} \) is a constant. The value of the auto-
covariance depends only on the time interval, \( t_j - t_i = \tau \), and is
denoted by \( C(\tau) \).

The experimental determination of the statistical quantities
expressed by [1] and [2] above is generally impractical, since a
large number of experimental runs must be performed before signifi-
cant statistical averages may be obtained. Fortunately, many
stationary random processes have the property that the time averages
along a single sample function are equal to the ensemble averages;
i.e., \( \overline{X(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-\tau/2}^{\tau/2} X(t) \, dt = \text{ave} \{ X(t) \} \)
\( C(\tau) = \overline{X(t)X(t+\tau)} = \lim_{T \to \infty} \frac{1}{T} \int_{-\tau/2}^{\tau/2} \left[ X(t) - \overline{X(t)} \right] \left[ X(t+\tau) - \overline{X(\tau)} \right] dt \)
This important property is known as ergodicity. Only stationary
random processes can be ergodic. If a random process is ergodic, all
of the relevant statistical averages can be performed in time, using
only one sample function, rather than averaging across the ensemble
of sample functions. Thus, the property of ergodicity results in
considerable economy of experimental effort.
Although an infinite set of statistical parameters, known as probability distributions, is required for a complete description of a random process, the Gaussian or normal process has the remarkable property (when stationary) that all of the probability distributions can be constructed from the autocovariance and the ensemble mean. Because many natural processes are at least approximately Gaussian, results obtained from the assumption of a Gaussian distribution are often used to provide reasonable estimates in analyses of processes for which the probability distribution is not known.

The analysis presented herein applies only to stationary ergodic random processes. If a time varying mean or a time dependent variance is present in the data, the process is nonstationary, and the following considerations are not applicable. One further simplification is made. It is assumed that the ergodic random process under consideration has a zero mean value. This assumption is not particularly restrictive, because processes with nonzero mean values can be easily treated by shifting the reference origin.

If the process is ergodic and has a zero mean value, the autocovariance can be

\[ C(\tau) = \overline{X(t)X(t+\tau)} = \lim_{m \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)X(t+\tau) \, dt \]  

[3]

If \( \tau \) is set equal to zero in [3], the statistical parameter known as the variance is obtained.

\[ \text{var} \{ X(t) \} = C(0) = \overline{[X(t)]^2} \]  

[4]
The positive square root of this quantity is called the root-mean-square. The function \( C(\tau) \) is sometimes referred to as the autocorrelation function. Herein, the term autocorrelation will be reserved for the normalized autocovariance \( C(\tau) C(0) \).

It may be shown that the autocovariance and the spectral density function are related through the Fourier transform.

\[
P(f) = \int_{-\infty}^{\infty} C(\tau) e^{i2\pi fr} d\tau \quad [5]
\]

\[
C(\tau) = \int_{-\infty}^{\infty} P(f) e^{i2\pi fr} df \quad [6]
\]

Equations [5] and [6] are the mathematical expression of the Wiener-Khintchine theorem which states that for a weakly stationary random process, the autocovariance and spectral density functions are related by Fourier transformations. For the present, [5] will serve as the definition of the spectral density function. An interpretation will be presented shortly. It may be further demonstrated that \( C(\tau) \) and \( P(f) \) are even functions, so that the expressions above can be written as Fourier cosine transforms.

\[
P(f) = \int_{-\infty}^{\infty} C(\tau) \cos(2\pi fr) d\tau \quad [7]
\]

\[
C(\tau) = \int_{-\infty}^{\infty} P(f) \cos(2\pi fr) df \quad [8]
\]

Although negative frequencies have no physical significance, the functions will be treated in the two-sided manner of [5] and [6] because the derivations are more easily developed in terms of the two-sided transforms.

The relationship between the physical spectrum (as measured by a spectrum analyzer, for example) and the two-sided spectrum is quite simple: the value of the physical spectral density, \( G(f) \), at \( f = f_0 \),
is equal to the sum of $P(f_1)$ and $P(-f_1)$. In short, $G(f_1) = 2P(f_1)$.

Refer to Fig. A-2 below.

Figure A-2.

Substituting $G(f) = 2P(f)$ into eq. [7] and [8], the relationships used in the practical computation of the spectral density function are obtained.

\[
G(f) = 4 \int_0^\infty C(r) \cos(2\pi fr) dr \quad [9]
\]

\[
C(r) = \int_0^\infty G(f) \cos(2\pi fr) df \quad [10]
\]

Another form of (9) and (10) results from normalizing the autocovariance with $C(0)$; the normalized expressions are then

\[
\frac{C(r)}{C(0)} = R(r) = \int_0^\infty \Phi(f) \cos(2\pi fr) df \quad [11]
\]

\[
\frac{G(f)}{C(0)} = \Phi(f) = 4 \int_0^\infty R(r) \cos(2\pi fr) dr \quad [12]
\]

As mentioned previously, the concept of negative frequencies is a mathematical artifice used to facilitate the derivations; without their introduction, complicated manipulations of trigonometric identities are required. In the sections which follow, the derivations will make use of the two-sided representation of the functions, and the examples will be illustrated with the physical spectral density function, designated by a $G$.  
Formally, the spectral density function is the Fourier transform of the autocorrelation function. An interpretation of the spectral density function may be obtained by setting $\gamma$ equal to zero in eq. [6].

$$C(0) = \overline{X(t)^2} = \int_{-\infty}^{\infty} P(f) df$$  \[13\]

Hence the mean square of the process is equal to the area under the spectral density function. From [13], one may interpret $P(f) df$ as the contribution to the variance from frequencies between $f$ and $f + df$. In a more physically appealing manner, one may also regard the spectral density function as a statistical decomposition of the variable in question with respect to frequency, indicating the relative intensities of the frequency components present. The spectral density function may also indicate the presence of periodic or quasi-periodic components in the sample function. To a structural engineer who is interested in knowing the characteristics of the exciting force on his design, the spectral density is an invaluable aid in determining the most probable frequencies the structure is likely to encounter under prototype operating conditions. For some cases, the spectral density of the exciting force can be used to predict the statistical characteristics of the structural displacements as well.

The expressions for the autocovariance and spectral density functions show that an infinite length of continuous record is required in order to calculate the necessary values. In practice, however, one has only a finite length of record, and if a digital
computer is used, the values of the finite record are known only at discrete points. Two important questions then arise: 1) What effects do the use of a finite length of record and discrete data have on the calculation of the spectral density function? 2) How reliable are the estimates which one is able to make? The first question will be answered by discussing the following four cases:

Case I  
An infinite length of continuous record will be treated briefly, in order to illustrate the ideal situation.

Case II  
The important effects of the use of discrete data on calculations will be brought out by treating the case of an infinite length of record digitized at equally spaced intervals.

Case III  
A finite length of continuous record will be examined to illustrate the effects of the finite length on estimates of the spectral density function.

Case IV  
The practical case involving a finite length of record with discrete, equally spaced data will be presented and discussed in detail.

Case I. Infinite Length of Record, Continuous Data

As indicated by the formal definition, if an infinite length of continuous record is available, the autocovariance function may be calculated from the relationship

\[
C(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)X(t+\tau)\,dt
\]
and the spectral density function may be obtained from

\[ P(f) = \int_{-\infty}^{\infty} C(\tau) \cos(2\pi \tau f) d\tau \]

Precisely how one accomplishes these steps is left to the reader’s ingenuity! Note, however, that the value of \( P(f) \) at each frequency, \( f \), in no way depends upon any other values of \( P(f) \). A unique value of \( P(f) \) may therefore be calculated for each \( f \), which shows that there is infinite resolution in the frequency domain.

**Case II. Infinite Length of Record, Values of the Function Known at Discrete Points**

Assume that one has an ergodic random process with a zero mean and that the autocovariance and spectral distribution function are known (i.e., the results for the infinite continuous case have somehow been obtained). Now suppose that the values of this exact autocovariance function designated by \( C(\tau) \) are taken at discrete points for \( 0, \pm \Delta \tau, \pm 2\Delta \tau, \pm 3\Delta \tau \), and so forth. These discrete points will be represented through the use of Dirac delta functions, \( \delta(\tau - g\Delta \tau) \).

The Dirac delta function (not a function in a rigorous mathematical sense) is defined such that \( \delta(x) = 0 \) if \( x \neq 0 \), and \( \int_{-\infty}^{\infty} \delta(x) dx = 1 \). Or, \( \delta(x - a) = 0 \) if \( x \neq a \), and \( \int_{-\infty}^{\infty} \delta(x - a) dx = 1 \). Hence \( \delta(\tau - g\Delta \tau) = 0 \) for \( \tau \neq g\Delta \tau \). To represent the autocovariance function \( C(\tau) \) which is known only at the discrete points \( \pm \Delta \tau, \pm 2\Delta \tau, \pm 3\Delta \tau \) and so on, a sequence of Dirac delta functions, called a Dirac comb, is introduced.

\[ D(\tau, \Delta \tau) = \Delta \tau \sum_{g=-\infty}^{g=\infty} \delta(\tau - g\Delta \tau) \]
Note that for any value, $r \Delta \tau$, only the term for $q = r \Delta \tau$ is nonzero. Hence, if the autocovariance function is to be represented in discrete form, one may write

$$C(r \Delta \tau) = \Delta \tau \sum_{q=-\infty}^{\infty} C(\tau) \delta(\tau - q \Delta \tau) = \nabla(\tau, \Delta \tau) C(\tau)$$

Figure A-3 is a pictorial representation of what has been accomplished by the mathematical expression above. Each point, $r \Delta \tau$, has been weighted by the value of the function, $C(\tau)$ at that point and the area of a rectangle as shown in Fig. A-3.

![Figure A-3](image)

It may be shown that the Fourier transform of a Dirac comb is

$$A(f, \frac{1}{\Delta \tau}) = \int_{-\infty}^{\infty} \nabla(\tau, \Delta \tau) e^{i 2\pi f \tau} d\tau = \sum_{q=-\infty}^{\infty} \delta(f - \frac{q}{\Delta \tau})$$

with the usual inverse representation

$$\nabla(\tau, \Delta \tau) = \int_{-\infty}^{\infty} A(f, \frac{1}{\Delta \tau}) e^{-i 2\pi f \tau} df$$

Observe that the Fourier transform of the infinite Dirac delta comb also yields an infinite Dirac comb in the frequency domain.

Since only the discrete data are available (although infinite in number for this case), a Fourier transform is suggested in order to calculate the spectral distribution function. Designating this
transform \( P_a(f) \) in order to distinguish it from the true spectrum \( P(f) \), there results

\[
P_a(f) = \int_{-\infty}^{\infty} V(\tau, \Delta \tau) C(\tau) e^{-i2\pi fr} d\tau
\]

Substituting the expression for \( V(\tau, \Delta \tau) \) into the above, one obtains

\[
P_a(f) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} A(f, \frac{1}{\Delta \tau}) e^{i2\pi fr} d\eta \right\} C(\tau) e^{-i2\pi fr} d\tau
\]

where \( \eta \) is a dummy variable. Rearranging the terms and the order of the integration the following results are obtained:

\[
P_a(f) = \int_{-\infty}^{\infty} A(f, \frac{1}{\Delta \tau}) \left[ \int_{-\infty}^{\infty} C(\tau) e^{-i2\pi (f-\eta)} d\tau \right] d\eta
\]

\[
= \int_{-\infty}^{\infty} A(f, \frac{1}{\Delta \tau}) P(f-\eta) d\eta
\]

This series of steps is known as convolution and may be written symbolically as

\[
P_a(f) = A(f, \frac{1}{\Delta \tau}) * P(f)
\]

Expanding this result, one obtains

\[
P_a(f) = \int_{-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left( \eta - \frac{q}{\Delta \tau} \right) P(f-\eta) d\eta
\]

which may be shown from the properties of the Dirac delta function to yield

\[
P_a(f) = \sum_{q=-\infty}^{\infty} P(f - \frac{q}{\Delta \tau})
\]

This extremely important result yields the relationship between \( P_a(f) \), which is all that can be calculated from the discrete data, and \( P(f) \), the true spectral density function for the ergodic random process under consideration. Several important points are:
1. If the true spectral density function, $P(f)$, is zero for $|f| > 1/(2\Delta \tau)$, then $P_a(f)$ is a periodic function with period $1/\Delta \tau$, as shown schematically in Fig. A-4.

![Figure A-4](image)

Figure A-4.

Rewriting eq. [14] so that the terms for $q < 0$, $q = 0$, and $q > 0$ are separated, there results:

$$P_a(f) = \sum_{q=-\infty}^{q=-1} P(f - \frac{q}{\Delta \tau}) + P(f) + \sum_{q=1}^{q=\infty} P(f - \frac{q}{\Delta \tau}) \quad [15]$$

Consider first the frequency range $|f| < 1/(2\Delta \tau)$. For $q \neq 0$, all of the frequencies $(f - 1/(2\Delta \tau))$ fall outside the range $-1/(2\Delta \tau) < f < 1/(2\Delta \tau)$. By hypothesis, $P(f) = 0$ for $|f| > 1/(2\Delta \tau)$. Therefore, the two summation terms in [15] are zero, and $P_a(f) = P(f)$ for $|f| > 1/(2\Delta \tau)$.

Next, consider $P_a(f)$ in the frequency range $1/(2\Delta \tau) < f < 1/(3\Delta \tau)$.

Because $|f| > 1/(2\Delta \tau)$, the term $P(f)$ in eq. [15] is zero. The summation term for $q = -\infty$ to $q = -1$ is also zero, since $|f - q/\Delta \tau| > 1/(2\Delta \tau)$. The only nonzero contribution comes from the summation term for $q > 0$. In fact, the only nonzero term occurs for $q = 1$. Then $|f - 1/(2\Delta \tau)| < 1/(2\Delta \tau)$ for $1/(2\Delta \tau) < f < 3/(2\Delta \tau)$ and $P_a(f) = P(f)$ in the interval...
\( \frac{1}{(2\Delta \tau)} < f < \frac{3}{(2\Delta \tau)} \). Likewise, for \( \frac{3}{(2\Delta \tau)} < f < \frac{5}{(2\Delta \tau)} \), the only contribution comes from the series term for \( q = 2 \). As \( q \) takes on all integral values from \(-\infty\) to \(+\infty\), a periodic function is generated with period \( 1/\Delta \tau \).

The function \( P_A(f) \), for \( |f| < 1/(2\Delta \tau) \), is called the principal part of \( P_A(f) \) and is designated by \( P_A(f) \). In conclusion, if the true spectral density function, \( P(f) \), is zero for \( |f| > 1/(2\Delta \tau) \), then the principal part, \( P_A(f) \), is identically equal to the true spectral density function, and nothing has been lost through the use of discrete data.

2. On the other hand, if the spectral density function, \( P(f) \), is not zero for \( |f| > 1/(2\Delta \tau) \), then contributions to \( P_A(f) \) will occur from the two summation terms in eq. [15]. Values of the true spectrum, \( P(f) \), will overlap in \( P_A(f) \) in the interval \( |f| > 1/(2\Delta \tau) \). Therefore, if proper precautions are not observed, the use of discrete data may yield completely erroneous results.

3. It is apparent from the foregoing that
\[
\int_{-\frac{1}{2\Delta \tau}}^{\frac{1}{2\Delta \tau}} P_A(f) \, df = \int_{-\frac{1}{2\Delta \tau}}^{\frac{1}{2\Delta \tau}} P_A(f) \, df = \int_{-\infty}^{\infty} P_A(f) \, df = \int_{-\infty}^{\infty} P(f) \, df
\]

Hence the area under the principal part of the spectrum, \( P_A(f) \), is the same as the area under the true spectral density function. This equality provides a check on numerical evaluation of \( P \), as will be pointed out later.

4. Since the concept of negative frequencies is only a mathematical artifice used to expedite derivations, rewriting eq. [15] for
positive frequencies makes the effects of discrete data much clearer.

Consider \( P_a(f') \) where \( 0 < f' < 1/(2\Delta \tau) \). Since \( P_a(f') \) and
\( P_A(f') \) are coincident for \( f' < 1/(2\Delta \tau) \), one obtains

\[
P_a(f') = P_A(f') = P(f') + \sum_{q=1}^{\infty} P(f' + \frac{q}{\Delta \tau}) + \sum_{q=-\infty}^{-1} P(f' + \frac{q}{\Delta \tau})
\]

Advantage can be taken of the fact that \( P(f) \) is an even function, i.e., \( P(f) = P(-f) \). One may write

\[
P_A(f') = P(f') + \sum_{q=1}^{\infty} \left[ P\left(\frac{q}{\Delta \tau} - f'\right) + P\left(\frac{q}{\Delta \tau} + f'\right) \right] \tag{15a}
\]

This relationship shows that the value of \( P_A(f') \) is the sum of the value \( P(f') \) plus all of the values \( P\left(\frac{q}{\Delta \tau} - f'\right) \) and \( P\left(\frac{q}{\Delta \tau} + f'\right) \), \( q = 1, 2, \ldots \). This phenomenon is known as aliasing. The frequencies in the true spectrum at \( (q/\Delta \tau - f') \) and \( (q/\Delta \tau + f') \), \( q = 1, 2, \ldots \) are said to be aliased at \( f' \) in \( P_z(f') \). The quantity \( P_z(f') \) is called the aliased spectrum. \( P_A(f') \) is called the principal alias.

Aliasing is sometimes referred to as spectrum folding. Fig. 5a shows the frequency axis for \( f > 0 \). Suppose that \( P(f_1), f_1 < 1/(2\Delta \tau) \) is to be calculated. The expression [15a] shows that the values of the true spectrum at frequencies \( (q/\Delta \tau - f_1) \) and \( (q/\Delta \tau + f_1) \) are all added algebraically. The net effect is to fold the frequency axis about the points \( q/(2\Delta \tau) \), \( q = 1, 2, \ldots \), much like a carpenter's folding rule, with the entire frequency axis from 0 to infinity folded into the interval from 0 to \( 1/(2\Delta \tau) \). The frequency of the first fold corresponds to \( f = 1/(2\Delta \tau) = f_N \) and is known as the Nyquist or folding frequency.
Example 1. Suppose that a sampling interval of $\Delta\tau = 2$ milliseconds is chosen. The folding frequency if $f_N = 1/(2\Delta\tau) = 250$ Hz. The value of the aliased spectrum at a frequency of 40 Hz is thus the sum of the values in the true spectrum at frequencies of 40 Hz., 460 and 540 Hz., 960 and 1040 Hz., 1460 and 1540 Hz., and so on.

A qualitative picture of aliasing may be obtained by referring to Fig. A-6. For a given sampling interval, $\Delta\tau$, all frequencies up to $f_N = 1/(2\Delta\tau)$ can be distinguished from each other. Frequencies higher than $f_N$ cannot be differentiated from those less than $f_N$. Fig. A-6 clarifies this statement. Which cosine wave is being sampled? Obviously either one could be—whence the phenomenon known as aliasing.
Aliasing is extremely important from a practical point of view. One of the principal uses of the spectral density function is to determine whether periodic components are present in the data. These components appear as peaks in the spectrum, as shown schematically in Fig. A-7a. If the choice of $\Delta \tau$ is such that the folding frequency is less than the frequency of the almost periodic component, the computed spectrum will be folded at the point $f = f_N$, and the peak will be aliased into the spectrum as shown in Fig. A-7b. The aliased spectrum thus yields a highly fallacious picture of the true spectrum. The presence of these peaks is truly a case of specters in spectra.

**Summary.** Given the exact autocovariance and spectral distribution functions, $C(\tau')$ and $P(f)$, obtained by some means from the ideal
analysis corresponding to Case I, the problem is to find the effects of using discrete data upon the analysis of the spectrum. Equally spaced values of the autocovariance are obtained at \( \tau = 0, \pm \Delta \tau \), and so on, and a Fourier transformation of the data is performed. The resulting relationships show that the frequencies \( |f| > 1/(2 \Delta \tau) \) in the true spectrum, \( P(f) \), are all folded or aliased into the interval \( |f| < 1/(2 \Delta \tau) \). The conclusion is that the use of discrete data causes aliasing. If the true spectral distribution is not very close to zero for \( |f| > 1/(2 \Delta \tau) \), the values in the aliased spectrum, \( P_A(f) \), will be significantly in error when compared with the true spectrum, \( P(f) \). The importance of the effects of aliasing cannot be overemphasized, and unless care is exercised in the choice of the sampling interval, \( \Delta \tau \), serious errors will result.

**Case III. Finite Length of Record, Continuous Data**

Suppose that a continuous record of finite length is to be analyzed. Let \( T_n \) be the total length of record and \( T_m \) be the maximum lag. From this sample record, one may compute the following:

\[
C_{oo}(\tau) = \frac{1}{T_n - |\tau|} \int_{(T_n - |\tau|)/2}^{(T_n - |\tau|)/2} X(t)X(t+\tau)\,dt
\]

If there are sufficient sample records of data, then

\[
\text{ave} \left[ C_{oo}(\tau) \right] = C(\tau)
\]

in the interval \( |\tau| < T_m \). Thus, estimates of the true autocovariance \( C(\tau) \) may be made with samples of finite, continuous records.

Because the limits of integration on the Fourier transform are \((-\infty, \infty)\), and because the function \( C(\tau) \) is defined only on the
finite interval $|\tau| < T_m$, the definition of the function $C(\tau)$ must be extended to infinity in order to perform the transform. To do so, the function $C_{oo}(\tau)$ may be multiplied by $D_1(\tau)$ which is chosen to be an even function of $\tau$ with $D_1(0) = 1$ and $D_1(\tau) = 0$ for $|\tau| > T_m$. The subscript, i, denotes the shape of the function $D_i(\tau)$ in the interval $|\tau| < T_m$. For purposes of illustration, $D_0(\tau)$ will be defined as the rectangular function

$$D_0(\tau) = \begin{cases} 1 & |\tau| < T_m \\ 0 & |\tau| > T_m \\ \frac{1}{2} & |\tau| = T_m \end{cases}$$

Writing $C_1(\tau) = D_1(\tau)C_{oo}(\tau)$, it is seen that $C_1(\tau)$ is defined for all values of $\tau$. The Fourier transform of this function is

$$P_1(f) = \int_{-\infty}^{\infty} D_1(\tau)C_{oo}(\tau)e^{-i2\pi fr}d\tau$$

Taking the ensemble average for a specific $f = f_1$, one obtains

$$\text{ave} \{P_1(f_1)\} = \int_{-\infty}^{\infty} D_1(\tau)C(\tau)e^{-i2\pi f_1\tau}d\tau$$

The Fourier transform of $D_1(\tau)$ is designated by $Q_1(f)$, and the inverse transform may be written as

$$D_1(\tau) = \int_{-\infty}^{\infty} Q_1(f)e^{i2\pi fr}df$$

$$Q_1(f) = \int_{-\infty}^{\infty} D_1(\tau)e^{i2\pi f\tau}d\tau$$

Performing convolutions there results

$$\text{ave} \{P_1(f)\} = \int_{-\infty}^{\infty} D_1(\tau)[\int_{-\infty}^{\infty} P(f)e^{i2\pi fr}df]e^{-i2\pi fr}d\tau = \int_{-\infty}^{\infty} Q_1(f_1-f)P(f)df$$

[16]
This last result shows that \( \text{ave} \{ P_i(f_1) \} \) is an average over frequency of the true spectrum, \( P(f) \), in frequencies "near" \( f_1 \). It is important to note that \( \text{ave} \{ P_i(f_1) \} \) does not give estimates of \( P(f_1) \) itself. The quantity, \( Q_i(f_1 - f) \), is a weighting function with integral effect of unity. One may look upon \( \text{ave} \{ P_i(f_1) \} \) as an impression of \( P(f) \) obtained through a window \( Q_i(f_1 - f) \). Therefore, \( Q_i(f) \) is called a spectral window corresponding to the lag window \( D_i(\gamma) \) (35).

**Summary.** The effect of a finite piece of record is to give "smoothed" spectral distributions, according to the particular weighting functions, \( Q_i(f) \), termed spectral windows. The infinite case gave infinite resolution, i.e., each value of \( P(f) \) was distinct. A finite piece of record yields averages over frequencies of \( P(f) \) in the neighborhood of the desired value, \( P(f_1) \). It is thus impossible to identify exactly the individual frequencies.

**Some Remarks About Spectral Windows.** This section discusses the concept of the spectral window in more detail, and shows how the lag window and spectral window may be used to obtain acceptable estimates of the spectral density function. Recall that in order to perform the Fourier transformation of the function, \( C(\gamma) \), defined only on the finite interval, \((-T_m, T_m)\), the range of definition of the function was extended to \((-\infty, \infty)\) by multiplying \( C(\gamma) \) by the function \( D_1(\gamma) \). Consider the function \( D_0(\gamma) \), which does nothing to the autocovariance except extend the definition of \( C(\gamma) \) (sometimes referred to as a "do nothing" lag window).
effect of the finite length of record, or equivalently, the finite range for \( C(\gamma) \), does affect the estimates of the spectral density function as [16] shows. Note carefully that the true spectrum, which one would like to determine, is multiplied by the function \( Q_o(f_1-f) \), and then summed from \(-\infty\) to \(\infty\). The estimate, 

\[ \text{ave} \left\{ P_o(f_1) \right\} \]

is called the smoothed estimate of \( P(f) \). Ideally, 

\( Q_o(f_1-f) \) should be a Dirac delta function, but as a little mathematics shows, this is precisely the result when the length of record is infinite. Therefore, the best estimates for the smoothed spectral density function should be obtained with a spectral window which is very peaked at \( f = f_1 \), and is nearly zero for frequencies other than \( f_1 \) (i.e., as close to a Dirac delta function as practicable).

The spectral window, \( Q_o(f) \), depicted in Fig. A–8, has a narrow, high peak, but the first side lobe, which has a peak value more than 20% of the main peak value, contributes significantly to the smoothing expressed by [16]. The large negative weights contributed by this first side lobe can lead even to negative estimates of the spectral density function, as a forthcoming example will show.

It is clear that a better shape for the spectral window, which means a different choice of lag window, would provide better estimates of the spectral density function. Thus far, cut and try has been the most successful procedure for finding suitable shapes for \( D_4(\gamma) \).

Two of these lag windows are

\[
D_2(\gamma) = \begin{cases} \frac{1}{2} & |\gamma| < \frac{T}{2} \\ 0 & |\gamma| > \frac{T}{2} \end{cases}
\]

\[
= \frac{1}{2} \left( 1 + \cos \left( \frac{\pi \gamma}{T} \right) \right) \quad |\gamma| < \frac{T}{2}
\]

\[
= 0 \quad |\gamma| > \frac{T}{2}
\]
the use of which is known as Hanning, and

\[ D_3(\tau) = 0.54 + 0.46 \cos(\frac{\pi \tau}{T_m}) \quad |\tau| < T_m \]

\[ = 0 \quad |\tau| > T_m \]

the use of which is known as Hamming.

Figure A-9 depicts the lag windows, \( D_2(\tau) \) and \( D_3(\tau) \), as well as the corresponding spectral windows, \( Q_2(f) \) and \( Q_3(f) \). Observe the change (10x) in vertical scale used to show the side lobes more clearly. A comparison of Fig. A-8 \( (Q_0(f)) \) with Fig. A-9 \( (Q_2(f) \) and \( Q_3(f) ) \) shows a marked improvement in side lobe characteristics. The maximum height of the first side lobe for the Hanning window is about 2.1\% of the main peak value, compared with a value for the height of the first side lobe of more than 20\% of the main peak value for \( Q_0(f) \). The price paid for this improvement in side lobe characteristics is a broadening of the main lobe.

The Hamming window has a much lower first side lobe than the Hanning window, but the height of the side lobes do not fall off as rapidly as do those for Hanning window. Other lag windows are given elsewhere (36,37). It is interesting to note that from the function, \( D_4(\tau)C_{oo}(\tau) \), which is a poor representation of \( C(\tau) \) itself, one may obtain very respectable estimates of the smoothed spectral density function.

Example 2. To clarify further the effects of a finite length of record on estimates of the spectral density, consider the auto-correlation function given by the following expression:

\[ R(\tau) = e^{-k\tau} \cos(2\pi f_0 \tau) \]
The exponentially damped cosine wave is an autocorrelation function characteristic of a narrow band random process (24), and was chosen for this example because the spectral density function can be obtained analytically.

\[
G(f) = 4 \int_0^\infty e^{-k\tau} \cos(2\pi f_0 \tau) \cos(2\pi f \tau) d\tau \\
= 2k \left[ \frac{1}{k^2 + (2\pi)^2 (f + f_0)^2} + \frac{1}{k^2 + (2\pi)^2 (f - f_0)^2} \right]
\]

A finite record produces the following autocorrelation function:

\[
R(\tau) = e^{-k\tau} \cos(2\pi f_0 \tau) \quad |\tau| < T_m
\]

To extend the range of this function to infinity and perform the Fourier transform, a suitable lag window must be chosen. For illustrative purposes, both \(D_0(\tau)\) and \(D_2(\tau)\) will be presented. Analytically, the smoothed spectral density functions for each of these two cases may be determined as follows. The autocorrelation function for the finite record is

\[
R_o(\tau) = D_0(\tau) e^{-k\tau} \cos(2\pi f_0 \tau)
\]

The spectral density function is computed as

\[
G_0(f) = 4 \int_0^\infty D_0(\tau) e^{-k\tau} \cos(2\pi f_0 \tau) \cos(2\pi f \tau) d\tau
\]

For the case with \(D_2(\tau)\) the steps are only outlined as follows:

\[
R_z(\tau) = D_2(\tau) e^{-k\tau} \cos(2\pi f_0 \tau) = \frac{1}{2} \left( 1 - \cos(\frac{\pi \tau}{T_m}) \right) e^{-k\tau} \cos(2\pi f_0 \tau)
\]

\[
G_z(f) = 4 \int_0^{T_m} \left( 1 - \cos(\frac{\pi \tau}{T_m}) \right) e^{-k\tau} \cos(2\pi f_0 \tau) \cos(2\pi f \tau) d\tau
\]

\[
G_z(f) = \frac{1}{4} G_0\left( f - \frac{1}{2T_m} \right) + \frac{1}{2} G_0(f) + \frac{1}{4} G_0\left( f + \frac{1}{2T_m} \right) \quad \text {[17]}
\]
The three functions, $G(f)$, $G_0(f)$, and $G_3(f)$, are plotted on Fig. A-10 for a choice of $k = 4$, $T_m = 0.5$ sec., and $f_0 = 20$ Hz. Because $G_2(f)$ can be expressed in terms of $G_0(f)$, $G_0(f + 1/2 T_m)$ and $G_0(f - 1/2 T_m)$ as given by (17), the functions, $G_0(f)$ and $G_2(f)$, were evaluated at points spaced $1/(2T_m)$ apart on the frequency axis. The "do nothing" lag window, $D_0(\tau)$ yields spectral estimates which oscillate about the true spectrum, $G(f)$. The undesirable large side lobes associated with $Q_0(f)$ are responsible for this behavior. Notice that from $R_2(\tau)$ which is a rather poor estimate of $R(\tau)$ itself, very respectable estimates of the spectral density function can be obtained. Except in the neighborhood of the peak, where a slight broadening and lowering of the peak value takes place, the agreement between the smoothed spectrum with the Hanning window, $Q_2(f)$, and the true spectrum, $G(f)$, is quite reasonable.

It should be noted that the frequency, $f_0$, is an integral multiple of $1/(2T_m)$. The estimates, $G_0(f)$, do not oscillate far from the true spectrum, $G(f)$. However, if the frequency, $f_0$, is not an integral multiple of $1/(2T_m)$, the presence of the large side lobes for $Q_0(f)$ can yield negative estimates of the spectral density. Table A-1 shows that the estimates, $G_0(f)$, can be negative for $f_0 = 19.75$ Hz. Note, however, that the spectral estimates obtained with $Q_2(f)$ are very reasonable estimates of the true spectrum, $G(f)$, in spite of the poor values of $G_0(f)$. 
Case IV. Finite Length of Record: Discrete Data

The preceding cases have been useful in separating and clarifying the effects of a finite length of record and digitized data upon analysis of the spectral density function. As the reader might suspect, these effects are combined in the practical situation discussed in this section. Consider again an ergodic random process, \( X(t) \). Suppose that this function is sampled at equally spaced intervals in time with \( \Delta t \) as the spacing. For a finite length of record, there will be \( n \) discrete points, from which the lagged products, \( C_r \), may be calculated for \( r = 0, 1, 2, \ldots, m \).

\[
C_r = \frac{1}{n-r} \sum_{q=0}^{q=n-r} x_q x_{q+r}
\]

Note that although there are \( n \) data points, there are only \( m+1 \) \( n \) values of \( C_r \). If several sample records are available, then treating these sample records as an ensemble, it may be shown that (35)

\[
\text{ave} \{ C_r \} = C(r \Delta t) = C(\tau)
\]

That is, the ensemble average of the computed autocovariances is equal to the value of the true autocovariance function. The \( m+1 \) values of \( C_r \) may be represented through the use of Dirac delta functions. Instead of the infinite Dirac comb used in Case II, a finite Dirac comb is required since there are only a finite number of \( C_r \)'s. Hence

\[
V_m(\tau, \Delta t) = \frac{\Delta t}{2} \delta(\tau + m \Delta t) + \sum_{q=-m}^{q=m-1} \delta(\tau - q \Delta t) + \frac{\Delta t}{2} \delta(\tau - m \Delta t)
\]
The weights at the end points are $\Delta \tau / 2$ because of the method used to represent the discrete data. The values of $C_\tau$ may be expressed in terms of the true autocovariance function, $C(\tau)$, as

$$\text{ave} \{ C_\tau \} = \int V_m(\tau, \Delta \tau) C(\tau \Delta \tau) d\tau$$

It is important to note that an ensemble operation is indicated. The computation of the autocovariances from the sample records also yields a random process. That is, from each sample record, a set of $C_\tau$ is obtained. The above expression states that the average value of the individual $C_\tau$'s is equal to the value of the discretized true autocovariance function. Applying the Fourier transform to obtain an estimate of the spectral density function for this case, there results

$$\text{ave} \left\{ P_{oA}(f) \right\} = \int_{-\infty}^{\infty} V_m(\tau, \Delta \tau) C(\tau \Delta \tau) e^{-i2\pi fr} d\tau$$

Again, the remarks concerning the ensemble average are applicable.

The subscript, o, refers to the shape of the spectral window associated with $V_m(\tau, \Delta \tau)$, and the subscript, A, refers to the fact that this spectral estimate is aliased.

For reasons which will appear shortly, $P_{oA}(f)$ will be evaluated only at frequencies $f = r/(2m \Delta r)$, $r = 0, 1, 2, \ldots, m$.

$$\text{ave} \left\{ P_{oA}(f, \Delta r) \right\} = \int_{-\infty}^{\infty} V_m(\tau, \Delta \tau) C(\tau \Delta \tau) e^{-i2\pi fr} d\tau$$

The relationships between the finite Dirac comb and its Fourier transform are

$$Q_\tau(f, \Delta r) = \int_{-\infty}^{\infty} V_m(\tau, \Delta \tau) e^{-i2\pi fr} d\tau$$

$$V_m(\tau, \Delta \tau) = \int_{-\infty}^{\infty} Q_\tau(f, \Delta r) e^{i2\pi fr} df$$
and performing convolution, one obtains for a particular \( f = r/(2m\Delta \tau) \)
\[
\text{ave} \{ P_{OA}(f) \} = \int_{-\infty}^{\infty} Q_0 \left( \frac{r}{2m\Delta \tau} - f, \Delta \tau \right) P(f) df
\]

Denoting \( P_{OA}(r/(2m\Delta \tau)) \) by \( V_r \), the above expression can be written
\[
\text{ave} \{ V_r \} = [ Q_0(f, \Delta \tau) * P(f) ]_{f = r/(2m\Delta \tau)}
\]

The spectral estimate, \( V_r \), may be regarded as a smoothed estimate of the true spectral density function, \( P(f) \), obtained by smoothing with the weighting function, \( Q_0(f, \Delta \tau) \).

As shown schematically in Fig. A-11, the finite Dirac comb may be represented as a product of the rectangular lag window, \( D_0(\tau) \), and the infinite Dirac comb, \( \nabla(\tau, \Delta \tau) \).

\[
\frac{V(\tau, \Delta \tau)}{D_0(\tau)} \quad \text{and} \quad \frac{V_m(\tau, \Delta \tau)}{}
\]

By applying the convolution to this function, the following result is obtained:
\[
Q_0(f, \Delta \tau) = Q_0(f) * A \left( f, \frac{1}{\Delta \tau} \right)
\]

Here \( Q_0(f) \) is the spectral window associated with \( D_0(\tau) \), as presented in the discussion of Case III, and \( A(f, 1/\Delta \tau) \) is the Fourier transform of \( \nabla(\tau, \Delta \tau) \), as presented in Case II. The function \( Q_0(f, \Delta \tau) \) can be interpreted as an aliased spectral window, denoted by \( Q_{OA}(f) \).
With the aid of some identities not presented herein (35), it may
be shown that
\[
\text{ave}\{v_r\} = \left[ Q_{\text{OA}}(f) \ast P_A(f) \right]_{f = r/(2m\Delta r)} = 0, 1, \ldots, m
\]

This expression provides the most useful interpretation of
\[ P_{\text{OA}}(r/2m\Delta r) = V_r. \]
The quantity \[ \text{ave}\{v_r\} \] may be regarded as
an estimate over frequency of the principal part, \[ P_A(f), \]
of the
aliased spectrum in the neighborhood of \[ f = r/(2m\Delta r), \]
as "seen
through the aliased spectral window, \[ Q_{\text{OA}}(f). \]
The aliased spectral
window, \[ Q_{\text{OA}}(f), \] has a shape very close to that of \[ Q(f), \]
for a suf-
ficiently large number of lags, \[ m. \]
In fact, as \[ m \]
approaches
infinity, \[ Q_{\text{OA}}(f) \] approaches \[ Q_o(f) \] (35).

Since \[ Q_{\text{OA}}(f) \] has undesirable side lobes, just as does \[ Q_o(f), \]
better spectral estimates can be obtained by using different spectral
windows. Two possible approaches will be discussed. For purposes
of illustration, the Hanning window will be used.

The first method involves multiplying the estimates of the auto-
covariance, \[ C_x, \] by the lag window corresponding to the spectral
window chosen. A Fourier transform of the resulting expression then
yields the spectral density estimates smoothed with the desired
spectral window.

A more convenient procedure consists of estimating values of
the spectral density function at frequencies spaced \[ 1/(2m\Delta r) \] apart.
After the raw estimates of the spectral densities, \[ V_r, \] have been
computed, the following smoothing operation is used:
\[ U_o = \frac{1}{2} V_0 + \frac{1}{2} V_r \]
\[ U_r = \frac{1}{4} V_{r-1} + \frac{1}{2} V_r + \frac{1}{4} V_{r+1} \]
\[ U_m = \frac{1}{2} V_{m-1} + \frac{1}{2} V_m \]

As shown elsewhere (35), the values obtained by this smoothing process are exactly equivalent to those of the first procedure outlined above. Other relationships similar to [18] are available for other choices of spectral windows (36,37). This smoothing after transformation (an easily programmable operation) requires that estimates of the spectral density function be spaced \(1/(2 \tau_m)\) apart, or in other words, that the values of \(V_r\) be computed at \(f = r/(2m \Delta \tau)\), \(r = 0, 1, 2, 3, \ldots, m\).

Example 3. Again consider the autocorrelation function given by the expression
\[ R(\tau) = e^{-k\tau} \cos(2\pi f_0 \tau) \quad |\tau| < T_m \]

To demonstrate the effects of aliasing, the values of the aliased spectrum given by the relationship
\[ G_o(\omega) = 4 \int_0^\infty D_0(\tau) e^{-k\tau} \cos(2\pi f_0 \tau) \cos(2\pi f \tau) d\tau \]
for the rectangular lag window and the values of the aliased spectrum for the Hanning lag window are plotted in Fig. A-12. As in the previous example, \(k = 4\), \(T_m = 0.5\), \(f_0 = 20\) Hz. The values of the aliased spectrum oscillate about the values obtained with the Hanning window, but because of aliasing, both estimates deviate from the true spectrum, \(G(\omega)\). The deviation is most apparent for \(f > 35\) Hz. Because the spectrum is folded at \(f = 50\) Hz. The estimate of the
spectral density must be in error at $f = 50$ Hz. by a factor of at least two. The agreement in the neighborhood of the peak is still reasonable, although a broadening and flattening of the peak again results for the smoothed estimates.

In this case, the values of the spectral density for $f = 50$ Hz. are small in comparison with the peak value, because the spectrum decreases in inverse proportion to $f^2$ for $f \gg f_o$. Therefore, the effects of aliasing are small in the neighborhood of the peak, but significant in the neighborhood of the folding frequency. It is clear that to obtain reasonable estimates of the spectral density, the folding frequency must be large enough that contributions from frequencies greater than $f_N$ are negligible.

The remarks of the previous example concerning the coincidence of $f_o$ as an integral multiple of $1/(2T_M)$ are applicable here also. As Table A-2 shows, if the frequency, $f_o$, is not an integral multiple of $1/(2T_M)$, then $G_{OA}(f)$ has some negative estimates, whereas the better estimates given by $G_{2A}(f)$ do not. It is clear that with the proper choice of lag window and folding frequency, the smoothed, aliased spectral density function is an acceptable estimate of the true spectrum, $G(f)$.

**Summary.** An analysis of a finite length of record known only at discrete, equally spaced points yields spectral estimates which are smoothed estimates of the principal part of the aliased spectrum. The spectral window also turns out to be an aliased spectral window. Aliasing is a consequence of using discrete data; smoothing is a consequence of a finite length of record. Because all practical
records are of finite length, some smoothing will always occur. Furthermore, if digital operations are used, aliasing cannot be avoided. It must be emphasized that both these effects be thoroughly understood before intelligent estimates of the spectral density function can be obtained.

Some Practical Relationships and Their Application

The preceding sections have dealt with the theoretical aspects of computing estimates of the spectral density function from discrete, equally spaced data. This section presents the practical application of the theoretical relationships which have been developed for Case IV, a finite length record, consisting of equally spaced data points.

Briefly, by way of review, the autocovariance and spectral density functions are related by Fourier transforms.

\[ P(f) = \int_{-\infty}^{\infty} C(\tau) e^{-i2\pi fr} d\tau \]

\[ C(\tau) = \int_{-\infty}^{\infty} P(f) e^{i2\pi fr} df \]

These two expressions may be reduced to

\[ P(f) = 2\int_{0}^{\infty} C(\tau) e^{-i2\pi fr} d\tau \]

\[ C(\tau) = 2\int_{0}^{\infty} P(f) e^{i2\pi fr} df \]

since \( P(f) \) and \( C(\tau) \) are even functions in \( f \) and \( \tau \) respectively.

Eliminating the artifice of negative frequencies and introducing the physically measurable spectrum \( G(f) \), one obtains

\[ G(f) = 4\int_{0}^{\infty} C(\tau) \cos(2\pi fr) d\tau \]

\[ C(\tau) = \int_{0}^{\infty} G(f) \cos(2\pi fr) df \]
As shown in the discussion of Case IV, a finite length of record
and discrete data lead to estimates of the spectral density func-
tion which are both smoothed and aliased.

\[
\text{ave} \{ P_a(f) \} = \int_{-\infty}^{\infty} V_n(\tau, \Delta \tau) C(\tau) e^{-i2\pi fr} d\tau
\]

Here, \( V_n(\tau, \Delta \tau) \) is the finite Dirac comb and \( C(\tau) \) is the true
autocoveriance function. It was also shown that

\[
V_n(\tau, \Delta \tau) = D_0(\tau) \cdot V(\tau, \Delta \tau) = D_0(\tau) \sum_{q=-\infty}^{q=\infty} S(\tau-q\Delta \tau)
\]

where \( V(\tau, \Delta \tau) \) is the infinite Dirac comb introduced in Case II.

Hence

\[
\text{ave} \{ P_a(f) \} = \int_{-\infty}^{\infty} D_0(\tau) \Delta \tau \sum_{q=-\infty}^{q=\infty} S(\tau-q\Delta \tau) C(\tau) e^{-i2\pi fr} d\tau
\]

and applying the property of the Dirac function that

\[
\int \delta(t-a)f(t)dt = f(a)
\]

one obtains

\[
\text{ave} \{ P_a(f) \} = \Delta \tau \left[ \sum_{q=-\infty}^{q=\infty} D_0(q\Delta \tau) C(q\Delta \tau) \cos(2\pi f q\Delta \tau) \right]
\]

The three functions \( D_0(q\Delta \tau) \), \( C(q\Delta \tau) \), and \( \cos(2\pi f q\Delta \tau) \) are all
even functions of \( \tau \). Recalling that \( D_0(\tau) \) is defined as

\[
D_0(\tau) = \begin{cases} 
1 & |\tau| < T_m \\
0 & |\tau| > T_m \\
1/2 & |\tau| = T_m
\end{cases}
\]

one may separate the series as follows:

\[
\text{ave} \{ P_a(f) \} = \Delta \tau \left[ \frac{1}{2} C(-m\Delta \tau) \cos(2\pi f(-m\Delta \tau)) + \sum_{q=-\infty}^{q=\infty} C(q\Delta \tau) \cos(2\pi f q\Delta \tau) \\
+ C(0) + \sum_{q=-\infty}^{q=\infty} C(q\Delta \tau) \cos(2\pi f q\Delta \tau) + \frac{1}{2} C(m\Delta \tau) \cos(2\pi f m\Delta \tau) \right]
\]
or, using subscript notation

\[ \text{ave} \left\{ P_{\Omega}(f) \right\} = \Delta f \left[ C_0 + 2 \sum_{q=1}^{q=m} C_q \cos(2\pi fcqf \Delta f) + C_m \cos(2\pi fcqf \Delta f) \right] \]

Because 2P(f) = G(f), the aliased spectrum may be expressed as

\[ \text{ave} \left\{ G_{\Omega}(f) \right\} = 2 \Delta f \left[ C_0 + 2 \sum_{q=1}^{q=m} C_q \cos(2\pi fcqf \Delta f) + C_m \cos(2\pi fcqf \Delta f) \right] \]

The function \( G_{\Omega}(f) \) is evaluated at points corresponding to

\[ f = r / (2m \Delta f) \], \( r = 0, 1, 2, \ldots, m \) so that the relationship for the

physically measurable (aliased) spectrum is

\[ V_r = \text{ave} \left\{ G_{\Omega}(f) \right\} = 2 \Delta f \left[ C_0 + 2 \sum_{q=1}^{q=m} C_q \cos \left( \frac{2\pi fr}{m} \right) + C_m \cos \left( \frac{2\pi fr}{m} \right) \right] \]

The estimates of the spectral density are spaced at an interval of

\[ 1 / (2m \Delta f) = 1 / (2T_m) \] apart on the frequency axis. A similar analysis

using the Hanning window yields the following:

\[ G_{\Omega}^{2A}(f) = \frac{1}{4} G_{\Omega} \left( f - \frac{1}{2T_m} \right) + \frac{1}{2} G_{\Omega} \left( f + \frac{1}{2T_m} \right) \]

Since results obtained with the Hanning window are much better than

the "raw" estimates given by \( G_{\Omega}(f) \), the relationship above indicates

one reason for evaluating the function \( G_{\Omega}(f) \) with a frequency

spacing of \( 1 / (2T_m) \).

As noted in the introductory paragraphs, it has been assumed

that the ergodic random process under consideration has a zero mean

value. There are few processes satisfying the conditions of

stationarity and ergodicity which also have a zero mean; nature is

seldom so obliging! Even if such a fortunate circumstance occurred,

there is a practical certainty that either the transducer used to

sense the physical variable or the supporting electronic equipment
would introduce a nonzero mean (i.e., a D.C. voltage).

It is therefore necessary to consider the effect of a nonzero mean value in the data and its influence on the computation of the autocovariance and spectral density function. The instantaneous value of the variable may be expressed as the sum of the mean value and the deviation from the mean.  \[ \bar{V} = \bar{V} + V' \]

The corresponding autocovariance is

\[ C(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} [\bar{V} + V'(t)][\bar{V} + V'(t+\tau)] \, dt \]

By definition

\[ \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} V'(t) \, dt = 0 \]

therefore

\[ C(\tau) = \bar{V}^2 + \bar{V}(t)V'(t+\tau) \]

The value of the autocovariance function thus approaches the square of the mean value of \( V(t) \) as \( \tau \) becomes very large. Intuitively, one may reason that correlation between the variables at time \( t + \tau \) and at time, \( t \), decreases as \( \tau \) increases. Since the value of the product \( v'(t) v'(t+\tau) \) has about a 50% probability of being positive or negative for large \( \tau \), one might expect that the mean value of this product would approach zero. For a truly random process (no periodic components), this is indeed the case, and

\[ C(\tau) \sim \bar{V}^2 \text{ as } \tau \to \infty. \]

The mean value appears in the spectral density function as a Dirac delta function located at \( f = 0 \) (sometimes visualized as a
zero frequency cosine wave). If it were possible to calculate the spectrum as outlined in Case I, then there would be no concern about the nonzero mean. However, practical considerations have shown that only a smoothed estimate of the spectral density function can be obtained. This smoothed estimate is calculated by weighting the entire true spectrum $P(f)$ with a weighting function called a spectral window. It has been noted, moreover, that all frequencies enter into this estimate. The presence of a large value of the spectral density for $f = 0$, which is the result of a nonzero mean value in the data, will thus influence the estimates of the spectral density for all frequencies. The effects of this bias, or leakage through the spectral window, are most pronounced at frequencies near $f = 0$; at higher frequencies, the effects are not as serious. One must be cautious, however. Although the side lobes of the spectral windows are small for frequencies far away from the main lobe, a small number multiplied by a large number may yield a result which can contribute significantly to the estimate of the spectral density.

Many applications require that the estimates of the spectral density at the low frequency end of the spectrum be as accurate as possible. It is therefore imperative that a correction be made to ergodic random processes with nonzero mean values. Two procedures for this correction are outlined below.

The simplest procedure for eliminating the mean value from the data is as follows:
1. Compute the mean value \( \bar{X}(t) = \frac{1}{N} \sum_{i=1}^{N} X_i \)

2. Compute a new set of data \( Y_i = X_i - \bar{X}(t) \)

3. Proceed with the computations of the autocovariance using the new set of data so determined.

This procedure presupposes that the data have been digitized, tabulated, and stored so that one has access to all of the original data points. Situations may arise with the use of on-line, real-time analysis in which it is impossible to store all of the data, if for no other reason than that their number becomes overwhelming and the housekeeping takes up too much time and memory storage space. In this case, eq. [19] suggests the following technique:

1. Compute the autocovariances. Simultaneously, keep track of the number of data points and accumulate a running sum.

2. Compute the mean value of the data from the acquired information, square this result, and then subtract it from each one of the computed autocovariances.

\[
C(\rho) = \bar{Y}^2 + \bar{V}(t)\bar{V}(t+\rho) - \bar{V}^2 = \frac{V'(t) V'(t+\rho)}{\sum_{i=1}^{N} X_i}
\]

The nonzero mean value has thus been effectively subtracted from the data.

For data digitized at equally spaced intervals, the procedure for obtaining the estimates of the spectral density function is as follows:
1. Compute the values of the autocovariances \( C_r \) from the expression

\[
C_r = \frac{1}{n-r} \sum_{q=r}^{q=n} x_q x_{q+r} \quad r = 0, 1, \ldots, m.
\]

In most practical applications, the digitized data will have a nonzero mean value. Therefore, either of the two correction procedures outlined above should be applied to the computed \( C_r \)'s.

2. If the normalized form of the spectrum is desired, compute the autocorrelation values from the autocovariances.

\[
R(\tau) = \frac{C(\tau)}{C(0)}
\]

3. The "raw" estimates of the spectral density function are computed from the relationship

\[
V_r = G_oA(\frac{r}{2m\Delta \tau}) = 2\Delta \tau \left[ C_0 + 2 \sum_{q=1}^{q=m-1} C_q \cos(\frac{r q \pi}{m}) + C_m \cos(r \pi) \right] \quad [20]
\]

or, for the normalized forms

\[
\Phi_r = \Phi(\frac{r}{2m\Delta \tau}) = 2\Delta \tau \left[ R_0 + 2 \sum_{q=1}^{q=m-1} C_q \cos(\frac{r q \pi}{m}) + C_m \cos(r \pi) \right]
\]

4. Smooth the "raw" estimates to obtain better values. If the Hanning window is used, the following expressions are applicable:

\[
U_0 = \frac{1}{2} V_0 + \frac{1}{2} V_1
\]

\[
U_r = \frac{1}{4} V_{r-1} + \frac{1}{2} V_r + \frac{1}{4} V_{r+1}
\]

\[
U_m = \frac{1}{2} V_{m-1} + \frac{1}{2} V_m
\]

\[
U'_0 = \frac{1}{2} \Phi_0 + \frac{1}{2} \Phi_1
\]

\[
U'_r = \frac{1}{4} \Phi_{r-1} + \frac{1}{2} \Phi_r + \frac{1}{4} \Phi_{r+1}
\]

\[
U'_m = \frac{1}{2} \Phi_{m-1} + \frac{1}{2} \Phi_m
\]

Other choices of spectral windows are described elsewhere (36,37).
A few remarks concerning eq. [20] are in order. At first glance, eq. [20] appears to be the simple trapezoidal rule for approximating an integral. It must be emphasized that this relationship has not been obtained by applying numerical techniques to the integral relationship given in eq. [9]. Quite the contrary, considerations of discrete data and a finite length of record have resulted in this form. Consequently, one cannot presume that the application of more sophisticated techniques of numerical integration will yield better results than the procedure outlined above. Again, understanding of the development of the relationships is the key to their successful application.

In closing this section, the dimensions of the spectral density function should be examined, since there are sometimes advantages to presenting the results in nondimensional form. The quantity \( R(\tau) \) is dimensionless, since it is the ratio of \( C(\tau) / C(0) \). Interestingly enough, \( \Phi \) is not dimensionless, as eq. [11] clearly reveals. Since \( R(\tau) \) and \( \cos 2\pi f \tau \) are both dimensionless, the dimensions of \( \Phi \) are those of \( d \tau \), or time. One possible choice of variables for making \( \Phi \) dimensionless is a representative velocity, \( U \), and a representative length, \( L \). The nondimensional axes corresponding to \( P(f) \) vs. \( f \) are therefore \( U \Phi / L \) and \( f L / U \). If these results are applied to the expression for \( R(0) \), eq. [11], one obtains

\[
\int_{0}^{\infty} \frac{U \Phi}{L} d\left(\frac{f L}{U}\right) = 1
\]
Thus, the area under the dimensionless spectral density function is unity, and is equal to the area under the principal alias of the dimensionless aliased spectral density function, which may be computed from eq. [20]. This result provides a numerical check on the computations for the estimates of the spectral density function.

So far, the question of what effects the use of a finite length of record and discrete data have on calculations for the spectral density function has been considered. The reliability of these estimates, an equally important question, will be treated in the following sections.

**Statistical reliability of the Spectral Estimates**

A thorough treatment of the reliability of the estimates of the spectral density function is somewhat beyond the scope of this Appendix. Rather than a detailed analysis, a general outline of the problem covering the essential points will be presented.

Intuitively, one might suspect that the reliability of the spectral estimates depends upon the reliability of the estimates of the autocovariances. The estimates of the autocovariances depend in turn upon the length of record, the number of lags, and the probability distribution of the variable, X(t). Since the probability distribution is not known, a normal or Gaussian distribution is usually assumed. This assumption yields results that are approximate if the variable is not normally distributed, but for purposes of planning the analysis and obtaining an estimate of reliability, this assumption will generally be entirely satisfactory.
In order to simplify the discussion, conditions as outlined in
Case III will be used. With a finite length of continuous record,
the effects of aliasing will not obscure the basic results. To
review, the autocovariance may be computed from
\[ C_{oo}(\tau) = \frac{1}{(T_n - 1) / 2} \int_{(T_n - 1) / 2}^{(T_n - 1) / 2} X(t)X(t+\tau)dt \]  \[ \text{[21]} \]
where \( T_n \) is the total length of the record, and \( \tau \) is the lag
distance. The Fourier transform of this expression extended to
cover the range \(-\infty\) to \( \infty \) by the lag window \( D_1(\tau) \) is
\[ P_1(f) = \int_{-\infty}^{\infty} D_1(\tau) C_{oo}(\tau) e^{-i2\pi f \tau} d\tau \]
From each sample record of the ensemble of possible sample records,
a different value of \( P_1(f) \) is obtained. The variable \( P_1(f) \) is
therefore a random variable. This fact requires a little shift in
thinking. The statistics of the random process \( P_1(f) \) provide the
answer to the problem of reliability. It has been already stated
that
\[ \text{ave} \{ P_1(f) \} = \int_{-\infty}^{\infty} D_1(\tau) C(\tau) e^{-i2\pi f \tau} d\tau \]
i.e., the average value of the random process, \( P_1(f) \), is the Fourier
transform of the true autocovariance function multiplied by the
lag window, \( D_1(\tau) \). This result forms the basis for the conclusions
drawn in discussing Case III. A measure of the relative
variability of a process is the coefficient of variation
\[ \sqrt{\text{var} \{ P_1(f) \}} \]
\[ \text{ave} \{ P_1(f) \} \]
From eq. [21] it is clear that as the maximum lag distance becomes larger, the length of record over which the integration is performed becomes smaller, and the estimates of $C_{oo}(\tau)$ become less reliable. Intuitively, one may conclude that $T_n$ should be a small fraction of $T_n$, and that there is an effective length of record used in the integration which is less than $T_n$. Blackman and Tukey give (35)

$$T'_n = T_n - \frac{1}{3} T_m$$

as the effective length for one sample record. If several sample records are available and the products $X(t)X(t + \tau)$ for the discrete case are all averaged without regard to which record they came from, then the effective length of record is

$$T'_n = \rho T_n - \frac{1}{3} T_m$$

After considerable mathematical manipulation, the following approximate result is obtained (35) for the coefficient of variation of $P_1(f)$:

$$\sqrt{\frac{T_m}{T'_n}}$$

Thus, as the effective length of record $T'_n$ becomes larger with respect to the maximum lag distance, $T_m$, the relative variance of the spectral estimates becomes smaller (e.g., more reliable). As long as the effects of aliasing are small, this result applies in the case of discrete data as well.

Equation [22] is only qualitative. In order to obtain a quantitative statement of reliability, the process $P_1(f)$ may be
compared to a standard process with known probability distribution.

Let \( z_1, z_2, \ldots, z_k \) be independent random variables with zero mean
value and a variance of unity.

\[
\text{ave} \{ Z_i \} = 0 \\
\text{var} \{ Z_i \} = 1
\]

By definition, the process

\[
\chi^2_k = Z_1^2 + Z_2^2 + \ldots + Z_k^2
\]

follows a chi-squared distribution with \( k \) degrees of freedom, where
\( k \) is the number of independent squares entering into the expression.

The quantity \( \chi^2_k \) is a positive quantity and the coefficient of
variation is \( \sqrt{2/k} \). Clearly as \( k \) increases, \( \chi^2_k \) becomes relatively
less variable.

Now \( P_1(f) \) is always a positive quantity and in the discrete
case is the linear sum of multiples of the autocovariances. Since
the autocovariances are the linear sums of pairs of products from
the sample record \( X(t) \), the autocovariances are approximately chi-
squared distributed. One may conclude that \( P_1(f) \) is approximately
chi-squared distributed. By comparing \( P_1(f) \) to some multiple of
which resembles \( P_1(f) \) in mean value and variance, the equivalent
degrees of freedom for \( P_1(f) \) are expressed as follows:

\[
\sqrt{\frac{T_m}{T_n'}} = \sqrt{\frac{2}{k}} \quad \text{or} \quad k = \frac{2T_n'}{T_m}
\]

A table for variables distributed as a fixed multiple of a chi-
squared distribution is presented below. The practical use of this
table will be illustrated by example.

Example. Suppose that a given choice of $T_n$ and $T_m$ yields 50 degrees of freedom. From Table A-3 the following statements can be made:

1. The probability is 0.8 that a single observed value will be in the interval 0.75 times the true mean value and 1.26 times the true mean value.

2. The probability is 0.9 that a single observed value will be in the interval 0.694 times the true mean value and 1.35 times the true mean value.

3. The probability is 0.95 that a single observed value will be in the interval 0.646 times the true mean value and 1.425 times the true mean value.

The reader familiar with statistics will recognize these statements as an interpretation of confidence intervals. Table A-3 may also be used as illustrated by the following example.

Example. How many degrees of freedom are required in order that one has 90% confidence that a single sample is within 10% of the true mean value? Consulting Table A-3, one sees that approximately 600 degrees of freedom are required.

Summary. An approximate analysis of the stability of the estimates of the spectral density function can be conveniently obtained by considering the equivalent degrees of freedom associated with multiples of chi-squared distributed variables. Since it is not necessary to obtain precise estimates on the reliability, and since
a small factor of safety is included in eq. [21], the values in Table I provide a reasonable statement of the confidence interval associated with the estimates of the spectral density function.
FIGURE A-8. SPECTRAL WINDOW $Q_o(f)$ AND THE CORRESPONDING LAG WINDOW, $D_o(T)$. 
FIGURE A-9. SPECTRAL WINDOWS $Q_2(f)$ AND $Q_3(f)$; CORRESPONDING LAG WINDOWS, $D_2(\tau)$ AND $D_3(\tau)$. 
FIGURE A-10. SPECTRAL DENSITY FUNCTIONS $G(\tau)$, $G_0(\tau)$, AND $G_2(\tau)$ FOR $R(\tau) = e^{-k\tau} \cos(2\pi \tau \cdot \tau)$. EXAMPLE 2.
Figure A-12. Spectral density functions $G(\xi)$, $G_{OA}(\xi)$, and $G_{2A}(\xi)$ for $R(\tau) = e^{-\lambda \tau} \cos(2\pi f \xi \tau)$. Example 3.
### TABLE A1  TABULAR VALUES OF THE SPECTRAL DENSITY FUNCTIONS

$G(f)$, $G_0(f)$ AND $G_2(f)$ FOR $f_o = 19.75$ Hz.

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<th>Frequency</th>
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<th>$G_0(f)$</th>
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TABLE A3

Distribution of Quantities Which are Distributed as a Fixed Multiple of Chi-Square. Ratios of Individual Values to Average Value Exceeded With Given Probabilities.

<table>
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<tr>
<th>Degrees of Freedom</th>
<th>Exceeded by 97.5% of all values</th>
<th>Exceeded by 95% of all values</th>
<th>Exceeded by 90% of all values</th>
<th>Exceeded by 10% of all values</th>
<th>Exceeded by 5% of all values</th>
<th>Exceeded by 2.5% of all values</th>
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<td>0.016</td>
<td>2.70</td>
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<td>0.105</td>
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<td>0.322</td>
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APPENDIX B

DESCRIPTION OF THE PROGRAM USED TO OBTAIN

THE AUTOCORRELATION FUNCTION
Appendix B. Description of the Program Used
To Obtain the Autocorrelation Function

Although the IBM 1801 Data Acquisition and Control System
installed at the Iowa Institute of Hydraulic Research can be programmed
in FORTRAN, it is not feasible to use this problem-oriented language
in real-time analysis. Real-time analysis requires that a large
number of individual samples be handled in a minimum amount of time
so that there will be sufficient data resolution and efficiency in
machine operation. The IBM 1801 is constructed with the electronic
circuits to perform integer arithmetic only. Floating point
arithmetic (manipulation of decimal numbers) must be carried out by
means of subroutines. Thus, each arithmetic operation in FORTRAN
requires the use of subroutines which are slow and inefficient in
comparison with "hardware" integer arithmetic. Furthermore, the
internal timers in the computer used to control the sampling of
analog voltages cannot be programmed with FORTRAN.

In order to program this computer efficiently for real-time
analysis, it is necessary to use Assembler Language. Assembler is
a powerful and highly flexible programming language, only one step
removed from machine language itself. The principal disadvantage is
the fact that each and every machine operation must be programmed,
and a great deal of "housekeeping" which the FORTRAN programmer never
need be concerned with, must be done. However, the flexibility of
the language, together with the close relationship to machine
language, permits many manipulations not possible in FORTRAN and fosters a clearer conceptual understanding of the machine operations. This in turn results in a more efficient and workable program.

Because the Assembler Language is machine oriented, a listing of the program will not be included herein. A discussion of the program logic in general terms, and of the choice of the important parameters necessary for statistical reliability will be more instructive. Figure B-1 depicts a flow chart of the program used to obtain the autocorrelation function. First, some housekeeping is performed, clearing storage locations and initializing others. Then the interval timer is started, and 1000 samples of the analog voltage are obtained at equally spaced intervals in time, $\Delta \tau$. During the time available between samples, the data for the mean and the mean square values are accumulated. After 1000 samples have been read in, the timer is shut off and the lagged products $x(t)x(t+\Delta \tau)$, $x(t)x(t+2\Delta \tau)$, ..., $x(t)x(t+m\Delta \tau)$ are computed. Because of the limited storage available in this system, another table of 1000 samples is obtained, the lagged products are accumulated, and added to those computed previously. The number of tables thus obtained is arbitrary; the program as finally developed collected 200 tables of 1000 samples. The 200,000 data points as treated in the program were required for statistical reliability. After the 200 tables have been processed, the data are corrected for the presence of a nonzero mean value, in accordance with procedures outlined in Appendix A, and the autocorrelation values are computed. The autocorrelation
data are then punched on cards to be used with a separate program for Fourier transformation to obtain the spectral density function.

Collection and processing of each set of data required approximately 31 minutes. During this time, the mean and the mean square values of the analog voltage from the experiment were kept constant, and were checked before and after each run.

The choice of the sampling interval, $\Delta \tau$, as well as of the number of tables, was determined by the amount of computer time available for analysis and the desired statistical reliability. Analog methods for obtaining the spectral density function showed measurable spectral density at frequencies up to 125 Hz. Blackman and Tukey (35) suggest that the folding frequency be twice the maximum frequency of interest. Therefore a folding frequency of 250 Hz. appeared reasonable. A sampling interval of $\Delta \tau = 1/(2f_n)$ or two milliseconds was selected.

It was felt that the estimates of the spectral density should be spaced less than 1/2 Hz. apart in frequency. The spacing, $f$, between estimates is given by

$$\Delta f = \frac{1}{2m \Delta \tau}$$

where $m$ is the number of points computed for the autocorrelation function. To satisfy the frequency spacing, $m$ must be greater than 500. As a first trial, 600 points were calculated. Previous experience with a computer program for evaluation of the mean square of the process showed that repeatable results could be obtained with about 200,000 data samples. Two-hundred tables were chosen for
processing. The effective length of record, as discussed in Appendix A, is given by

$$T'_n = \rho \left( T_n - \frac{1}{3} T_m \right)$$

where $T_n$ is the total length of one piece of record, $T_m$ is the maximum lag interval equal to $m$, and $\rho$ is the number of pieces of record.

For 200 tables and $T_m = 600 \times 0.002 = 1.2$ seconds, the value of $T'_n$ is 320 seconds. The equivalent number of degrees of freedom (refer to Appendix A) is computed from

$$k = \frac{T'_n}{T_m}$$

From Table A-3, the estimates of the spectral density for a 90% confidence interval lie within 0.90 and 1.10 of the true value. In other words, the probability is 90% that the estimate of the spectral density will be within plus or minus 10% of the true value. Because of the various assumptions used in deriving the relationships for statistical reliability, the only sure test is to repeat the experiment several times and examine the data scatter. Repetition of the experiments for the present investigation indicated reasonable repeatability. The program as developed with a sampling interval of two milliseconds and processing two-hundred tables of data was judged adequate with respect to statistical reliability and computation time.

All of the previous spectral analyses performed at the Iowa Institute of Hydraulic Research used a commercially available analog spectrum analyzer manufactured by Intercontinental Instruments.
Incorporated. This instrument was used initially by Locher (7), and subsequently by Tatinclaux (5), Chu (9), and others. A comparison of the analog technique with the digital procedure makes a very interesting contrast. If the form of the spectral density function is not known, several repeated analog spectra are necessary to define the function. The first run determines the general shape of the curve, and indicates the regions where peaks are located. In order to define these peaks more precisely, a second and sometimes a third experimental run is required. Finally, the low-frequency range is analyzed with the aid of a tape recorder as explained by Tatinclaux (5). Thus, at least three runs are necessary, each run requiring three to four hours, depending on the behavior of the electronic equipment. Upon completion of the experiment, the raw data must be analyzed, reduced, and plotted, a procedure which consumes about two hours for each run. The analog approach requires approximately six hours for data acquisition and analysis for each spectrum.

With the digital computer, the autocorrelation function can be obtained in thirty-one minutes, with the output punched on cards. A computer program transforms the autocorrelation function into estimates of the spectral density in twenty-five minutes. The data are smoothed and printed in a form suitable for plotting. The total time required is about 1.5 hours, or about one-fourth the time for the analog method. Furthermore, the analog technique provides only twenty-five to thirty points per run, whereas the digital computer
produces six-hundred points evenly spaced over the frequency range from 0 to 250 Hz. In this particular case, the digital analysis of the data considerably reduced the time and effort necessary for spectrum analysis, and provided far more information and more reliable data than that attainable with the analog approach. The extensive application of spectrum analysis to the fluctuating forces measured in this study simply would not have been feasible without the aid of the digital computer.
START

Clear storage locations for products

Start timer

Collect 1000 samples of data spaced $\Delta T$ apart in time

Using the time available between samples accumulate the data for the mean and the mean square

Stop the timer

Compute the products $\frac{X(t)X(t+\Delta T)}{X(t)X(t+2\Delta T)}$ through $\frac{X(t)X(t+m\Delta T)}{X(t)X(t+2m\Delta T)}$

If 200 tables have been processed, proceed, if not, go back and process another table of 1000 samples and accumulate the mean products

$\beta$

FIGURE B1. FLOW CHART FOR THE EVALUATION OF THE AUTOCORRELATION FUNCTION
Correct the data for a non-zero mean value

Compute the values of the auto-correlation function and punch the results on cards

STOP
APPENDIX C

DIMENSIONLESS REPRESENTATION OF THE
INPUT-OUTPUT RELATIONSHIP
FOR THE RESPONSE OF A LINEAR SYSTEM
WITH ONE DEGREE OF FREEDOM
APPENDIX C. DIMENSIONLESS REPRESENTATION OF THE
INPUT-OUTPUT RELATIONSHIP FOR THE RESPONSE OF
A LINEAR SYSTEM WITH ONE DEGREE OF FREEDOM

The mean square value of the displacement of the mass, \( m \), in
terms of the statistical properties of the random process, \( P(t) \),
representing the fluctuating force is given by the expression

\[
\overline{\chi'^2} = \int_{-\infty}^{\infty} \frac{|H(f)|^2}{|H(0)|^2} |P_r(f)|^2 df
\]

where

\[
|H(f)|^2 = \frac{1}{m^2 (2\pi f_0)^2 \left[ \left( \frac{Z_{\text{eff}}}{2\pi f_0} \right)^2 + 4 \frac{\sigma^2}{2\pi f_0} \right]} - \frac{1}{m^2 (2\pi f_0)^2 \left[ \left( \frac{Z_{\text{eff}}}{2\pi f_0} \right)^2 + 4 \frac{\sigma^2}{2\pi f_0} \right]}
\]

Both \( |H(f)|^2 \) and \( P_r(f) \) are even functions of the frequency, \( f \). Hence

\[
\overline{\chi'^2} = \int_{0}^{\infty} |H(f)|^2 2P_r(f) df
\]

If both sides of the expression above are multiplied by \( 1/|H(0)|^2 \) and
\( 1/F'^2 \) and the expression under the integral is multiplied by

\[
f = \frac{U_0}{b}(b/\omega)
\]

one obtains

\[
\overline{\chi'^2} = \int_{0}^{\infty} \frac{|H(f)|^2}{|H(0)|^2} \left[ \frac{U_0}{b} \frac{2P_r(f)}{F'^2} \right] \frac{df}{f}
\]

Noting that \( |H(0)|^2 = k^2 \) where \( k \) is the spring constant, and utilizing
the result from Appendix A that \( 2P_r(f)/F'^2 = \Phi \), a nondimensional
form for the mean square displacement is

\[
\frac{\overline{\chi'^2}}{k^2 F'^2} = \int_{0}^{\infty} \frac{|H(f)|^2}{|H(0)|^2} \frac{U_0}{b} \Phi(f/\omega) \frac{df}{f}
\]

This expression can be evaluated by numerical methods, where
is a typical spectrum of the fluctuating force as measured in the
present investigation.
The spectral density function of the displacement of the mass, \( m \), is given by

\[
\mathcal{P}_x(f) = |H(f)|^2 \mathcal{P}_F(f)
\]

Multiplying both sides of the above expression by \( \frac{U_b^2}{\overline{F}^2 x^2} \) and then multiplying the right-hand side by \( f \cdot |H(0)|^2 / |H(f)|^2 \) one obtains

\[
\frac{1}{\overline{F}^2} \left[ \frac{2 \mathcal{P}_x(f)}{x^2} \right] \frac{U_b}{b} = \left[ \frac{|H(f)|^2}{|H(0)|^2} \right] \frac{2 \mathcal{P}_F(f)}{\overline{F}^2} \frac{U_b}{b} \frac{k^2}{x^2} \]

\[
\frac{U_b}{b} \Phi_x \left( \frac{f b}{U_b} \right) = \frac{U_b}{b} \Phi_F \left( \frac{f b}{U_b} \right) \frac{|H(f)|^2}{|H(0)|^2} \left[ \frac{k^2}{\overline{F}^2} \frac{1}{x^2} \right] \quad [B1]
\]

The last quantity in brackets on the right-hand side of [B1] is the reciprocal of the expression derived for the determination of the nondimensional mean square value of the displacement. Straightforward numerical methods, again using a typical spectrum of the fluctuating force, were used in the evaluation of the spectrum of the displacement given by [B1].