058:268 Turbulent flows
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Handout: Large Eddy Simulation II

Dynamic Smagorinsky model

Idea: Calculate model parameter from smallest resolved scales

- Similar to doing a priori parameter estimation on the fly
  - Treating LES velocity field as exact (or as DNS field)
- Technique
  - Filter LES velocity as in an a priori test
  - Use wider filter (test filter) than the original LES filter (grid filter) $\widetilde{\Delta} > \Delta$
    - *Usually $\widetilde{\Delta} \sim 2\Delta$
  - Assume same model applies at both levels with:
    - * Different length scale but same model parameter
- Compute model parameter from simulation itself
  - Then use it to compute eddy viscosity field in the simulation
- No external information needed
- Method is self-contained and self-consistent or dynamic
- Gives variation of parameter with time, spatial location $C_d = C_d(x, y, z, t)$

Recall general form Smagorinsky model:

$$\tau_{ik} = -\overline{u_i u_k} + \bar{u}_i \bar{u}_k = 2\nu_t \bar{S}_{ik} \qquad (1)$$

$$\nu_t = C_d \Delta^2 \left|\bar{S}\right|$$

$$\left|\bar{S}\right| = \sqrt{2\bar{S}_{ik}\bar{S}_{ik}}$$

In dynamic model $C_d = C_d(x, y, z, t)$.

From equation (1), the stress in the Smagorinsky model can be rewritten as:

$$\tau_{ik} = 2C_d \alpha_{ik} \qquad (2)$$
$$\alpha_{ik} = \Delta^2 \left|\bar{S}\right| \bar{S}_{ik}$$

- The main question is what should be the value of C_d?
- We know that the optimum value varies from flow to flow and that its value should be reduced near solid walls to reduce the amount of dissipation introduced by the SGS model.
- Can we obtain a model that can do that automatically?

A basic introduction to the idea of dynamic modeling

Consider an arbitrary nonlinear term t(u), which a is known function of the field variables, u, and suppose we wish to determine its filtered value by modeling the subgrid residual with an algebraic model m(u), which depends on the field variables but, in general, can also depend explicitly on space and time and on other parameters such as the filter width Δ. The value of the filtered term is then the sum of the filtered and modeled parts:

$$\overline{t(u)} = t(\bar{u}) + m(\bar{u}) \qquad (3)$$

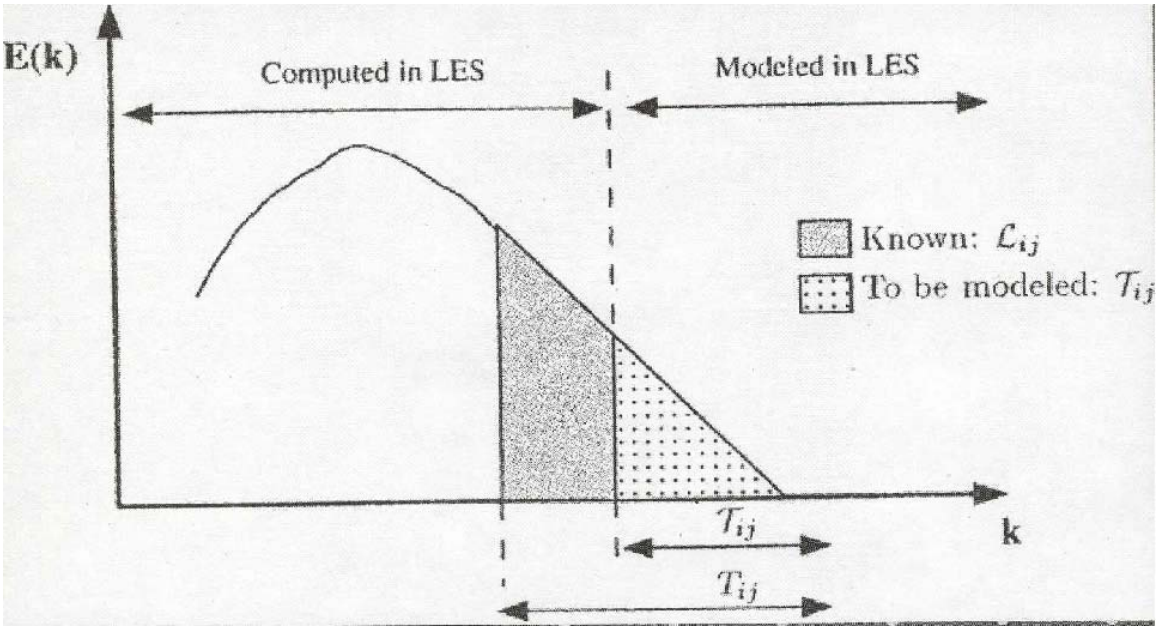
The basic idea behind the basic procedure is to consider how t(u) and m(u) vary with the filter width. In particular, an expression similar to the previous one for the value of the filtered term at a larger filter width, referred as test filter '~', can be written as:

$$\overline{\tilde{t}(u)} = t(\tilde{\bar{u}}) + m(\tilde{\bar{u}}) \qquad (4)$$

If relation (3) is test filtered and subtracted from (4) we get:

$$\tilde{\tilde{\tilde{t}}}(\bar{u}) - \tilde{t}(\bar{u}) = \tilde{m}(\bar{u}) - \tilde{\tilde{\tilde{m}}}(\bar{u}) \qquad (5)$$

Remarkably, **all the terms in this equation are computable from the resolved field**. It represents the 'band-pass filtered' contribution to the nonlinear term in the scale range between the grid and the test filter levels. A consistent SGS model should contribute the same amount as the resolved field in this band. The key to the dynamic procedure is to use this identity as a constraint for calibration of SGS models. Note that while (5) is an exact identity when m(u) is the exact subgrid residual, it should only be expected to hold in a statistical sense when m(u) is modeled.



Let's choose $t=u_iu_k$ and $m(\bar{u})$ a model for the stress τ_{ik} as in equation (2). Remember that by definition the subtest stress T_{ik} is just τ_{ik} with '$\bar{}$' replaced by '$\tilde{\bar{}}$', ($T_{ik} = \tilde{\tilde{\tilde{u_iu_k}}} - \tilde{\bar{u}}_i\tilde{\bar{u}}_j$). The test filter width is taken normally larger than the width of the test filter (typically twice). So from (5) we get:

$$\widetilde{\overline{u}_i\overline{u}_k} - \widetilde{\overline{u}}_i\widetilde{\overline{u}}_k = T_{ik} - \widetilde{\tau}_{ik} \qquad (6)$$

in our case $T_{ik} = 2C_d\beta_{ik}$ with $\beta_{ik} = \widetilde{\Delta}^2\left|\widetilde{\overline{S}}\right|\widetilde{\overline{S}}_{ik}$, similar to α_{ik} in equation (2). Relation (6) tells us that there is a relation between the modeled stresses at two different filter levels and the resolved stresses. **Equation (6) is also called Germano's identity**. Upon introduction of the proposed expressions for T_{ik} and τ_{ik}, (6) can be satisfied only approximately. The other important observation is that the left hand side of (6) is just the resolved stress denoted $-L_{ik}$.

$$-L_{ik} \approx 2C_d(\beta_{ik} - \widetilde{\alpha}_{ik}) \qquad (7)$$

or

$$L_{ik} \approx 2C_d\Delta^2 M_{ik} \qquad (8)$$

where

$$L_{ik} = -\widetilde{\overline{u}_i\overline{u}_k} + \widetilde{\overline{u}}_i\widetilde{\overline{u}}_k \qquad M_{ik} = \left(\widetilde{\Delta}/\Delta\right)^2\left|\widetilde{\overline{S}}\right|\widetilde{\overline{S}}_{ik} - \widetilde{\left|\overline{S}\right|\overline{S}_{ik}} \qquad (9)$$

To obtain equation (7), an **additional assumption** was made, that **the model parameter can be removed from the filter** as the test filter acts on $C_d\alpha_{ik}$. This is equivalent that assuming that C_d is locally constant, which is not true.

Thus

$$C_d\Delta^2 \approx \frac{1}{2}\frac{L_{ik}}{M_{ik}} \qquad (10)$$

Equation (10) contains in fact a system of 6 independent equations that have to be used to determine only one constant. Lilly proposed to normalize (10) by multiplying both L_{ik} and M_{ik} by M_{ik} and sum over both i and k indices. The final expression is averaged over the homogeneous directions in the flow (if they exists) to improve the robustness of the predictions (the coefficient C_d should be positive for stability reasons, something that is not guaranteed by the dynamic procedure).

$$C_d \Delta^2 = \frac{1}{2} \frac{< L_{ik} M_{ik} >}{< M_{ik} M_{ik} >} \tag{11}$$

where <> means summation over all points in the homogeneous direction.
In a more general context and to be more consistent, a least-square procedure can be used to minimize the error e_{ik} in (7). So all what one has to do is to minimize:

$$E^2 = e_{ik} e_{ik} .$$

$$e_{ik} = L_{ik} + 2C_d (\beta_{ik} - \tilde{\alpha}_{ik}) = L_{ik} - 2C_d \Delta^2 M_{ik} \tag{12}$$

$$E^2 = L_{ik}^2 - 4C_d \Delta^2 L_{ik} M_{ik} + 4C_d^2 \Delta^4 M_{ik} M_{ik} \tag{13}$$

To minimize the error we require $\frac{dE^2}{dC_d} = 0$. So

$$C_d \Delta^2 M_{ik} M_{ik} = L_{ik} M_{ik} \tag{14}$$

Thus one obtains same expression for $C_d \Delta^2$ as in (11) but in a more consistent way that can be used, as we are going to see later, for more complex models, including multiple coefficients models (e.g., mixed models).

Another important observation based on results from numerical simulations of various flows is that the model coefficient has large variations in fairly small regions of the flow. For instance, in simulations of (decay of) isotropic turbulence:

$$\left\langle C_d^2 \right\rangle - \left\langle C_d \right\rangle^2 \approx 10 \left\langle C_d \right\rangle^2 \qquad (15)$$

which suggests that the removal of C_d from the filter in equation (7) was not justified.

Dynamic Smagorinsky model: Advantages

- Self-contained; no need to specify parameter
- Inexpensive; adds 10-15% to cost (vs. constant coefficient Smagorinsky)
- Removes some of the problems associated with constant coefficient Smagorinsky model but needs method to stabilize simulation
- Eliminates need to prescribe length scale
  - No need to choose formula for Δ with anisotropic grid
  - Dynamic method actually computes ν_t rather than C_d
    - *If Δ changed, C_d changes to compensate
- Predicts zero eddy viscosity in laminar regions of the flow
- No need for near-wall correction
  - Gives proper near-wall behavior automatically
  - No need for wall-damping functions
- No need to modify for 'extra strains'
  - Stratification, rotation effects automatically included
    - *Reduces C_d where Richardson number high
- Applied successfully to many flows, for example
  - Homogeneous flows, simple free shear flows
  - Rotating flows, stratified flows, atmospheric boundary layer
  - Backward facing step, rib, diffuser
- When it works well, it is self-compensating
  - Suppose that there is too much energy in smallest resolved

* Dynamic model will increase eddy viscosity

* Result is reduction of energy in small scales later

- Opposite behavior if energy is too small

Dynamic Smagorinsky model: Disadvantages

- Parameter variation too large
  - Variance is ten times the mean (see above, eqn. 15)
  - Produces large negative values of ν_t

*Can be negative for long time, over sizeable region

*Results in numerical instability

Reasons for poor behavior:

- Assumption of constant C_d in equation (7) is incorrect
  - Should not remove from filter
- Too much reliance on smallest scales
  - Not accurately simulated, noisy
- Rapid variation in both space and time of parameter and eddy viscosity (including the presence of negative values)
  - causes numerical instability
  - average value reasonable
  - negative values something like back-scatter

* may be useful, but dangerous to include if too large

Simple way of reducing variation:

*average over space (local or over homogeneous direction) and/or time

* can average ν_t

* additionally filter (smooth) the predicted field of C_d

- Cutoff negative values of parameter ($\nu+\nu_t \geq 0$)

*Called clipping

More advanced SGS models

There are several ways to improve the deficiencies of the dynamic Smagorinsky model, especially for nonhomogeneous flows when averaging in one or two directions that would reduce the sharp fluctuations in the values of the model coefficient is not possible.

A very consistent approach is the **dynamic localization model** of Ghosal et al. (1995) in which an integral equation is solved to determine the model coefficient. There is no need to average expressions locally or in the homogeneous directions but an integral formulation of the identity (6) or (7) was used. This identity rigorously removed the mathematical inconsistency (the fact that $\widetilde{\widetilde{C_d \alpha_{ik}}}$ is not exact equal to $C_d \widetilde{\alpha}_{ik}$) at the expense of solving an integral equation at each time step (this is computationally quite expensive, comparable to the solution of a Poisson equation). The integral equation is obtained by minimizing a functional (in this case the integral of the error, see equation (12), over the entire domain). If no additional constraints are imposed on C_d, the integral equation is linear. If the additional constraint that $C_d>0$ is imposed everywhere in the flow the integral equation is nonlinear, thus more expensive to solve.

Several simpler variants exist that reduce the computational overhead related to solving exactly the integral equation, but then one cannot mathematically guaranty that $C_d>0$. Piomelli and Liu proposed that instead of solving directly the integral equation for C_d, one can try to solve it iteratively. At each new time step one has to make a guess for C_d^{n+1}; call it C^* (simplest possibility $C^* = C_d^n$). One can substitute this value into the integral term and evaluate it. Then determinate the new value of C^* and iterate until convergence. It was shown that the modified localized model works well for a number of flows (channel, bluff bodies) and computational cost is not much higher than the standard dynamic model.

Another approach proposed by Meneveau et al. (1996) for non-homogeneous flows is to use **Lagrangian averaging**, meaning to take the average in (11) over a path line (back in time).

The use of a **mixed model** (discussed later) that includes a scale-similarity part will decrease the contribution of the Smagorinsky part and decrease significantly the level of spuriously high values of the Smagorinsky coefficient (aside from allowing backscatter effects to be correctly accounted for and in a numerically robust way).

Finally, **solving an additional transport equation for the SGS turbulent kinetic energy** that allows us to determine the turbulent fluctuations scale (the total amount of energy in the subgrid scales, k) has also been shown to produce better distributions for C_d. This approach will be discussed later.

The Dynamic Lagrangian SGS Model (Meneveau et al., 1996)

This model tries to improve the performance of the dynamic Smagorinsky model for non-homogeneous flows.

Rationale

- Dynamic model not reliable without averaging
  - Numerical instability
- So some averaging necessary
  - Global averaging successful but requires homogeneous direction
  - Local averaging possible but results depend on volume chosen
- Need an averaging procedure that works in complex flows

Define Lagrangian averaging <> as:

$$I_f =< f >= \int_{-\infty}^{t} f(t')W(t-t')dt' \qquad (16)$$

The idea behind using Lagrangian averaging in the formula for the model coefficient

$$C_d\Delta^2 = \frac{1}{2}\frac{< L_{ij}M_{ij} >}{< M_{ij}M_{ij} >} = -\frac{1}{2}\frac{I^n_{LM}}{I^n_{MM}} \tag{17}$$

is based on the consideration that memory effects should be calculated in a Lagrangian framework, following the fluid particle, rather than in an Eulerian framework, which sees different particles, with different histories, at each instant. Thus the integral in (16) is carried out following a fluid path-line, W(t) is an exponential weighting function chosen to give more weight to recent times, n denotes the time step and, using (16):

$$I_{LM} = \int_{-\infty}^{t} L_{ij}(t')M_{ij}(t')W(t-t')dt'$$

$$I_{MM} = \int_{-\infty}^{t} M_{ij}(t')M_{ij}(t')W(t-t')dt' \tag{18}$$

To simplify the numerical implementation (where it is not easy/computationally efficient to integrate to far back in time) one can choose the weight function:

$$W(t) = 1/T \cdot \exp(-t/T) \tag{19}$$

with the time constant defined as:

$$T = 1.5\Delta \cdot \left(-8I^n_{LM}I^n_{MM}\right)^{-1/8} \tag{20}$$

in which case one can show that the integrals (18) can be approximated by:

$$I^n_{LM}(\vec{x}) = H\left\{\varepsilon L^n_{ij}M^n_{ij} + (1-\varepsilon)I^{n-1}_{LM}(\vec{x}-\vec{u}^n\Delta t)\right\}$$

$$I^n_{LM}(\vec{x}) = H\left\{\varepsilon L^n_{ij}M^n_{ij} + (1-\varepsilon)I^{n-1}_{LM}(\vec{x}-\vec{u}^n\Delta t)\right\} \tag{21}$$

where H is the ramp function, the coefficient ε is defined as:

$$\varepsilon = \frac{\Delta t / T}{1 + \Delta t / T} \tag{22}$$

and the evaluation of the integrals (21) at $\vec{x} - \vec{u}^{\,n}\Delta t$ can be performed by linear interpolation. In order to avoid complex values for T (see equation 20), if $C_d(\vec{x}, t) = 0$ is reached, I_{LM}^{n} is set to zero. Of course, an extra parameter T that characterizes the averaging time was introduced. Ideally the averaging time should depend on the local time scale.

It was found that this way of estimating the model coefficient reduced the fraction of the points where the model will predict negative values and reduces the variability in the dynamic coefficient values with the effect of finally improving the robustness of the numerical simulation. This model can also be combined with mixed models.